A graphic representation of traction for a 3D elastic/plastic stress state

Zhubin He (corresponding author), Z. R. Wang and Bugang Teng
School of Materials Science and Engineering, Harbin Institute of Technology, Harbin 150001, People's Republic of China
E-mail: hithe@263.net

Abstract The stress ellipsoid is often used to describe the variation scope of total stress of one stress state. But the direction of one vector from the original point to the ellipsoid surface is not always the one of the corresponding acting plane. A 3D graphic representation of total stress for an elastic/plastic stress state is given in this paper for the first time. The magnitude of the total stress on one plane can be described by the projection along the corresponding normal direction. The real direction of the total stress can be given directly. In a principal axes system, the figure is symmetric about the three principal axes. As for a plastic stress state, the stereograph of total stress has its own characters because the yield criterion must be obeyed. From the total stress figure, it is easy to get the magnitude of total stress acting on a certain plane. It is very helpful in understanding the concept of the stress ellipsoid and the variation of total stress across different acting planes.

Keywords scientific visualization; stress ellipsoid; total stress

Introduction

Scientific visualization is currently a very active and vital area of research, teaching and development. The success of scientific visualization is mainly due to the soundness of the basic premise behind it, that is, the basic idea of using computer-generated pictures to gain information and understanding from data (geometry) and relationships (topology). This is an extremely intuitive and very important concept, which is having a profound and widespread impact on the methodology of science and engineering [1].

Scientific visualization is often used in scientific computing and engineering analysis. It can also be used to give intuitive expression and in-depth understanding of many complex theories and equations, especially in mathematics analysis, physics and mechanics.

On a traditional elasticity/plasticity course, the relationship between a stress state and the associated traction and its classification are often a challenge to students. For any stress state, the total stress varies with the normal directions of different acting planes. The stress ellipsoid is often used to describe the variation envelope of total stress [2–4]. But in the stress ellipsoid, one vector from the original point to the surface can give only the magnitude of total stress. The direction of the vector is not that of the corresponding acting plane, except for the three principal axes. The vector on the ellipsoid surface will present both the magnitude and the direction only at a hydrostatic stress state. For this reason, a three-dimensional figure of total stress for one stress state is given in this paper for the first time.

The magnitude of the total stress on one plane is described by the projection along the corresponding normal direction, and its real direction can be given directly. This is consistent with the traditional stress ellipsoid. If we rotate all the vectors in the total stress figure to its real directions, we will get the corresponding traditional stress ellipsoid. But from such a plot of total stress, it is very easy to find the magnitude of total stress acting on a certain plane.

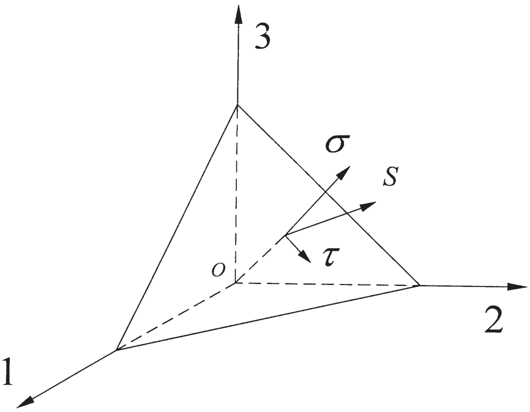

Fig. 1 *Stresses on an arbitrary plane in the principal axes system.*

Equations

Fig. 1 shows the total stress, S, normal stress, σ, and shear stress, τ, on the plane with direction cosines l, m, n in a three-dimensional principal axes system. As shown in Fig. 1, the direction of the total stress, S, is different from the normal direction of the plane. The two directions will be the same only when the shear stress, τ, equals zero.

The components of the total stress in the three principal directions, i.e., S_1, S_2, S_3, can be derived from the equilibrium equations [2]:

$$S_1 = l \cdot \sigma_1 \tag{1.1}$$

$$S_2 = m \cdot \sigma_2 \tag{1.2}$$

$$S_3 = n \cdot \sigma_3 \tag{1.3}$$

where σ_1, σ_2, σ_3 are the three principal stresses, respectively.

For $l^2 + m^2 + n^2 = 1$, the equations above can be rewritten as:

$$\frac{S_1^2}{\sigma_1^2} + \frac{S_2^2}{\sigma_2^2} + \frac{S_3^2}{\sigma_3^2} = 1 \tag{2}$$

Equation 2 describes the so-called stress ellipsoid, as shown in Fig. 2 [3]. From Fig. 2, we can see the variation scope of the total stress at one given stress state. In Fig. 2, S is one point on the ellipsoid surface, and the vector $\overrightarrow{OS}$ presents the total stress of

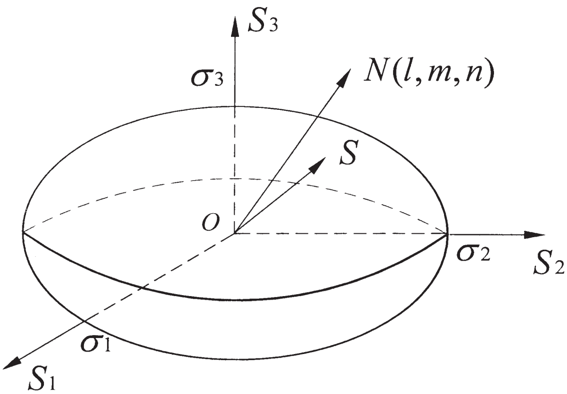

Fig. 2 *The stress ellipsoid.*

the plane with normal direction $\overrightarrow{ON}$. It can be seen that $\overrightarrow{OS}$ and $\overrightarrow{ON}$ are generally not the same because of the nonzero shear stress.

The total stress on the plane with direction cosines l, m, n can be given as:

$$S = \sqrt{S_1^2 + S_2^2 + S_3^2} = \sqrt{l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2} \tag{3}$$

3D figure of total stress for one stress state

For one stress state, σ_1, σ_2, σ_3, the total stress, S, in the direction with three direction cosines l, m, n, can be given from equation (3). Draw a vector along the direction of $\overrightarrow{ON}$ with the module equal to S, and this vector will present the magnitude of the total stress in this direction. With all the l, m, n available, we can get the vectors in all directions. Connect all the ends of the vectors by MATLAB [5], and a stereograph of total stress can be given.

Fig. 3 shows all the stress figures (on the left) and the corresponding stress ellipsoids (on the right) of typical stress states. In these figures, we assume $\sigma_1 > \sigma_2 > \sigma_3$, and the data in the brackets are the values of σ_1, σ_2, σ_3, respectively. S_1, S_2 and S_3 are the three principal axes, i.e., the direction of the principal stresses σ_1, σ_2, σ_3.

From Fig. 3, it can be seen that the 3D figure of total stress is very different from the stress ellipsoid. The total stress figure is not ellipse shaped. It is generally locally depressed in the principal direction with minimal magnitude.

The shapes are symmetric about the three principal axes, which can also be seen from equation (3). In equation (3), the magnitude of total stress, S, is determined by the square of l, m, n. That means the total stresses on two planes symmetric to each other about one or more axes will have the same magnitude.

Fig. 3(a) and Fig. 3(b) are similar in shape, but different in size. The shape of the total stress figure reveals the relative proportion of the magnitude of the three principal stresses. The size of the shape is determined by the magnitude of the three principal stresses.

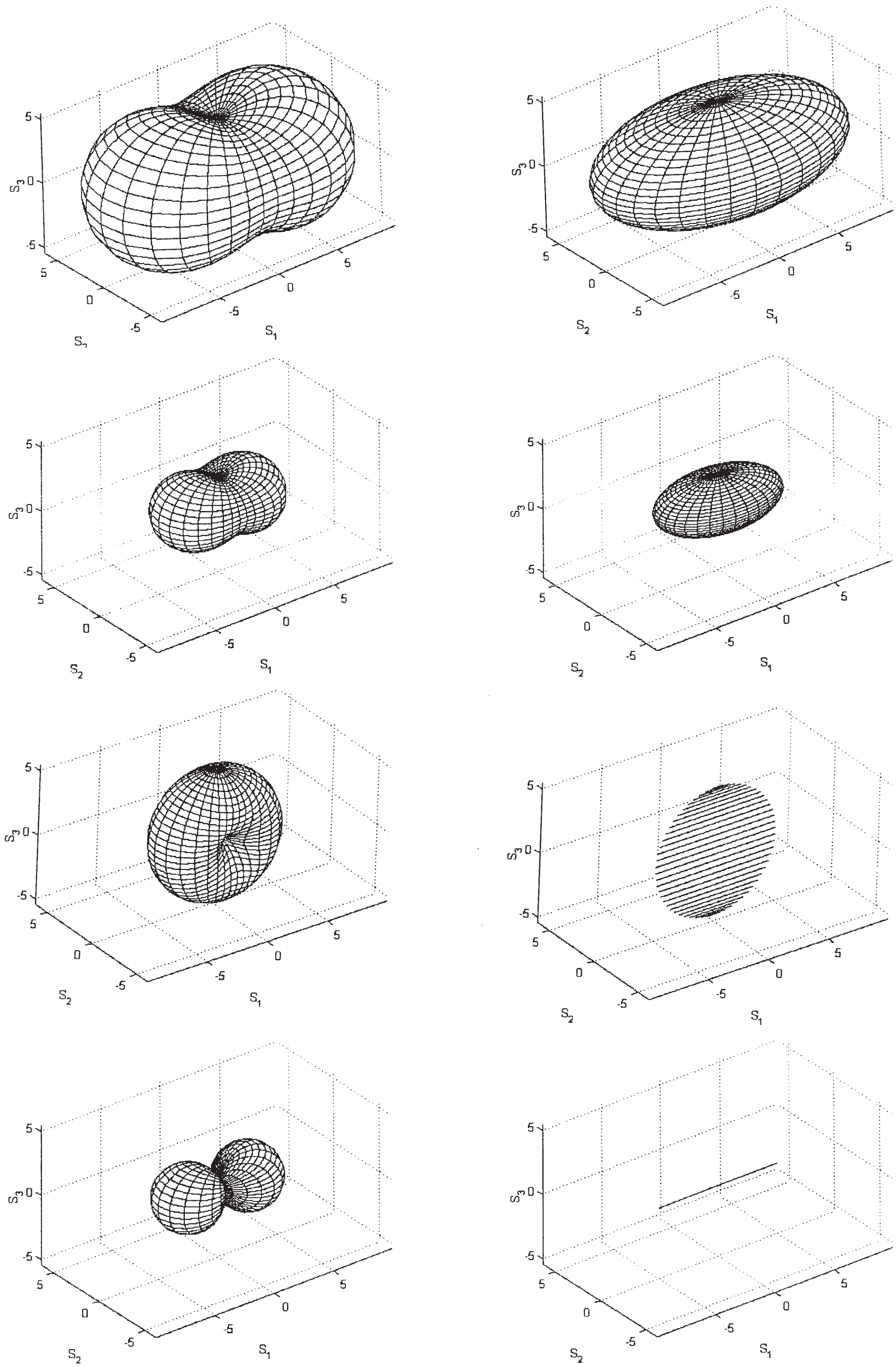

Fig. 3 *Total stress figures (left) and stress ellipsoids (right) of typical stress states. The values of σ_1, σ_2, and σ_3 are, respectively: (a) [10, 6, 4], (b) [5, 3, 2], (c) [5, 0, –5], (d) [5, 0, 0].*

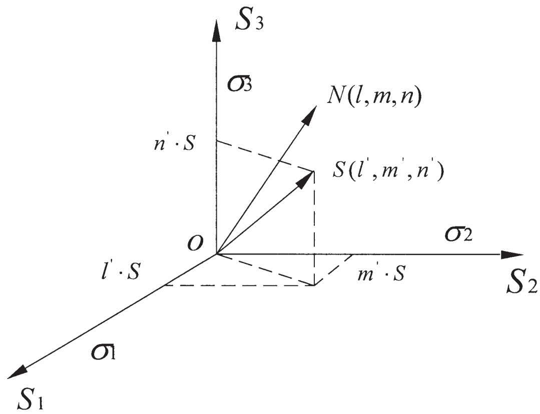

Fig. 4 *Total stress in the principal axis system.*

In Fig. 3(c), the total stress, S, in the direction of S_2 is equal to zero, because $\sigma_2 = 0$. The stress ellipsoid becomes an ellipse.

In Fig. 3(d), all the total stresses in the plane S_2OS_3 are equal to zero because there is only one non-zero principal stress. The stress ellipsoid becomes a line.

When $l = \pm1$, $m = 0$, $n = 0$, then $S = \sigma_1$. The direction of the total stress is along S_1, and the intercept of the total stress figure with axis S_1 equals the magnitude of σ_1. In the same way, we can also arrange the intercepts of the total stress figure such that S_2 and S_3 are equal to the magnitude of σ_2 and σ_3, respectively. So, as discussed above, from the total stress figure, we can know the magnitude of the three principal stresses. In this respect, the total stress figure is consistent with the stress ellipsoid.

The direction of total stress

As discussed above, the direction of the vector from the origin to the surface, as shown in Fig. 2, does not correspond to the total stress described by the stress ellipsoid. It is necessary to find the relations between the total stress figure and the stress ellipsoid.

Fig. 4 shows the total stress, S, on the plane with direction cosines l, m, n. Assume l', m', n' are the direction cosines of vector $\overrightarrow{OS}$.

Then the components of S along the three principal axes are:

$$S_1 = l' \cdot S \tag{4.1}$$

$$S_2 = m' \cdot S \tag{4.2}$$

$$S_3 = n' \cdot S \tag{4.3}$$

From equations (1) and (4), we get:

$$l \cdot \sigma_1 = l' \cdot S \tag{5.1}$$

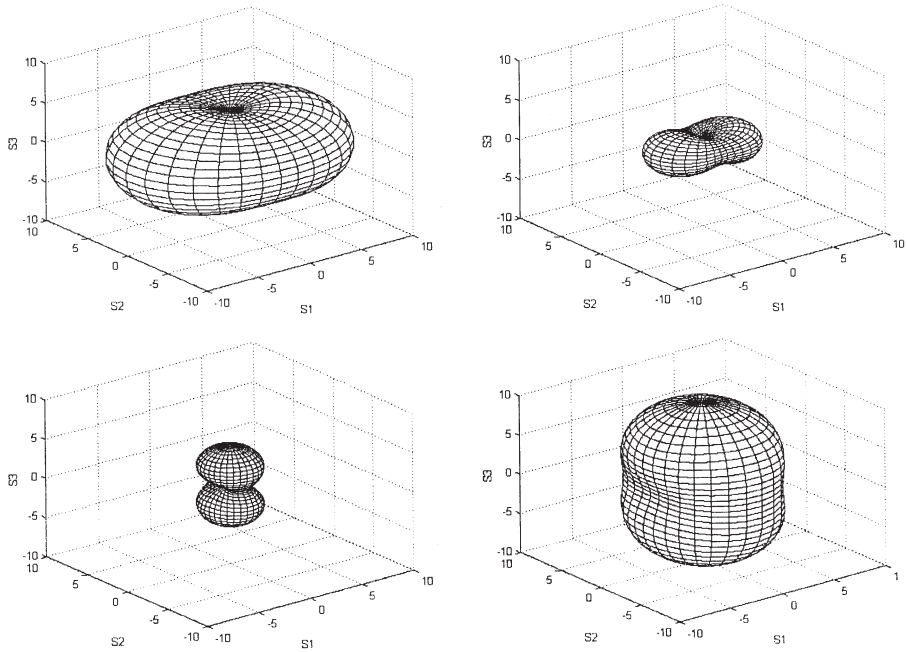

Fig. 5 *The total stress figures under plastic conditions. The values of σ_1, σ_2, and σ_3 are, respectively: (a) [10, 8, 5], (b) [5, 3, 0], (c) [0, –2, –5], (d) [–5, –7, –10].*

$$m \cdot \sigma_2 = m' \cdot S \tag{5.2}$$

$$n \cdot \sigma_3 = n' \cdot S \tag{5.3}$$

And then:

$$l' = l \cdot \sigma_1 / S \tag{6.1}$$

$$m' = m \cdot \sigma_2 / S \tag{6.2}$$

$$n' = m \cdot \sigma_3 / S \tag{6.3}$$

Equation (6.1–6.3) gives the real direction of the total stress acting on a certain plane with direction cosines *l*, *m*, *n*.

If all the vectors in the total stress figure are rotated to the direction given by equation (6.1–6.3), we can get the corresponding traditional stress ellipsoid.

3D total stress figure under plastic conditions

Assume $\sigma_1 > \sigma_2 > \sigma_3$, then Tresca's criterion can be written as $\sigma_1 - \sigma_3 = \sigma_Y$. Varying σ_1 and σ_3 under plastic conditions, i.e., $\sigma_1 - \sigma_3 = \sigma_Y$ is satisfied, and a series of total stress figures can be given. Fig. 5 shows several figures under plastic conditions. The yield stress is set as $\sigma_Y = 5$.

From Fig. 5, it can be seen that the figures of total stress under plastic conditions are very different. The total stress figure of a one-sign stress state (i.e., $\sigma_1 > \sigma_3 > 0$ or $0 > \sigma_1 > \sigma_3$) is larger than that of a two-sign stress state (i.e., $\sigma_1 > 0$ and $\sigma_3 < 0$).

Conclusions

(1) A graphic representation of traction for one stress state is given in this paper. The total stress figure is very different from the stress ellipsoid. It is generally locally depressed in the principal direction with minimal magnitude, and not ellipse shaped. It is also symmetric about the three principal axes.

(2) The shape of the total stress figure reveals the relative proportion of the magnitude of the three principal stresses. The size of the figure is determined by the magnitude of the three principal stresses. The real direction of the total stress is given by equation (6.1–6.3).

(3) The total stress figure is consistent with the stress ellipsoid. If all the projections drawn from the original point to the surface are rotated to its real direction given by equation (6.1–6.3), we get the traditional stress ellipsoid. But from the total stress figure, it is very easy to find the magnitude of total stress acting on a certain plane. It is very helpful in understanding the concept of the stress ellipsoid and the variation of total stress across different acting planes.

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