
Avenues for computing the heat transfer from a circular fin of hyperbolic cross-sectional profile

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Abstract The circular fin of hyperbolic profile with constant thermal conductivity and uniform convective coefficient is of remarkable importance in heat transfer engineering. This particular fin resembles the optimal circular fin of convex parabolic profile, which delivers the maximum heat transfer for a given volume of material. The circular fin of hyperbolic profile is governed by a two-term differential equation of second order with one variable coefficient which, via a transformation, can be converted into a Bessel equation. Setting aside the use of Bessel functions, the present paper addresses three elementary computational procedures that are amenable to heat transfer education: the power series method, the finite-difference technique and the shooting method. With the second technique, the system of algebraic equations was solved by the elimination of unknowns, or the Gauss elimination method. With the third method, a fourth-order Runge–Kutta integration algorithm was paired with a linear interpolation formula for solving the system of two differential equations of first order. The three computational procedures are capable of producing approximate temperature distributions and approximate heat transfer rates of high quality. All analytical and numerical calculations were carried out on a personal computer.

Keywords circular fin; hyperbolic profile; power series; finite differences; shooting

Notation

c	normalized radii ratio, r_1/r_2 , dimensionless
h	convective coefficient, $\text{W}/\text{m}^2\text{K}$
H^2	thermogeometric parameter, $h(1/\delta_1 r_1)/k$, $1/\text{m}^3$
$H_\nu^{(*)}$	modified Bessel function of second kind and order ν
$I_\nu^{(*)}$	modified Bessel function of first kind and order ν
k	constant thermal conductivity, W/mK
M^2	dimensionless H^2 or modified Biot number, $H^2 r_2^3$
Q	heat transfer rate, W
Q_i	ideal heat transfer rate, W
r	radial coordinate, m
r_1	inner radius, m
r_2	outer radius, m
R	dimensionless r , r/r_2
T	temperature, $^\circ\text{C}$
T_b	base temperature, $^\circ\text{C}$
T_t	tip temperature, $^\circ\text{C}$
T_∞	fluid temperature, $^\circ\text{C}$
w	length, $r_2 - r_1$, m

$y(R)$	profile function, m
z	dimensionless temperature gradient, $d\theta/dR$
δ_1	inner semi-thickness, m
δ_2	outer semi-thickness, m
η	fin efficiency, Q/Q_i , dimensionless
θ	dimensionless T , $(T - T_\infty)/(T_b - T_\infty)$

Introduction

If there is a large difference in convective coefficients between two sides of a heat exchanger, the transfer area on the side with the lower convective coefficient is often enlarged by the addition of fins. Circular fins usually increase the heat transfer from primary metallic surfaces (e.g., tubes, cylinders and rods) to surrounding fluids. A common type of fin used in heat exchanger tubes is the circular fin of constant or tapered thickness. Typical applications of circular fins are found in air-cooled engines of motorcycles and automobiles, and in tubes of liquid–gas heat exchangers used in the refrigeration industry.

An inspection of the sections devoted to fin heat transfer in all textbooks on heat transfer reveals two contrasting scenarios. A superabundance of information exists for straight fins of constant or tapered thickness. Some authors even report efficiencies of straight fins with five different cross-sectional profiles. In contrast, however, coverage of circular fins of tapered thickness is scarce. Surprisingly, only two authors, Lienhard [1] and Mills [2], include the expression for the efficiency of the circular fin of hyperbolic profile in their textbooks, but without further explanation. Interestingly, the importance of the circular fin of hyperbolic profile stems from the fact that its cross-sectional profile is very close to the cross-sectional profile of the optimal fin of convex parabolic profile discovered by Schmidt [3]. The latter is cited in Schneider [4].

The present paper focuses on educational aspects of fin heat transfer. Thus, the main objective is to delineate three computational procedures for solving the descriptive equation for a circular fin of hyperbolic profile: a Bessel equation. The three computational procedures are the power series method, the finite-difference technique and the shooting method. As shown below, the easiness of the three computational procedures is instrumental in facilitating the approximate numerical determination of the temperature curves and fin efficiencies, without having to use Bessel functions. In a broader perspective, the paper seeks the integration of knowledge from two courses in the mechanical engineering curriculum, namely numerical methods and heat transfer.

The analytical and numerical computations were carried out with the symbolic algebra software Maple [5] on a personal computer.

Modeling

A circular fin is bounded by two symmetric hyperbolas, as depicted in Fig. 1, such that four dimensions mold the fin: inner radius, r_1 , inner semi-thickness, δ_1 , outer

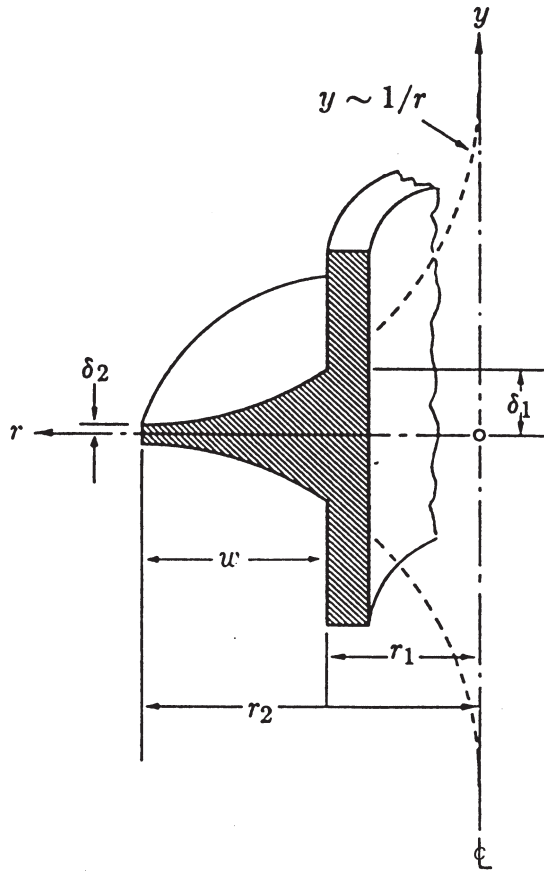


Fig. 1 Sketch of a circular fin of hyperbolic profile.

radius, r_2 , and outer semi-thickness, δ_2 . The circular fin of hyperbolic profile resembles the optimal fin of convex parabolic profile discovered by Schmidt [3]. The optimal fin delivers the maximum heat transfer for a given volume of material [3, 4].

When the cross-section of a fin is variable, the formulation of the problem involves a linear second-order differential equation with variable coefficients. Under the assumption of constant thermal conductivity, k , and uniform convective coefficient, h , the transfer of heat from a circular fin of hyperbolic profile to a surrounding fluid at a temperature T_∞ is modeled by the dimensionless fin equation [4]:

$$R^2 \frac{d^2\theta}{dR^2} - M^2 R^3 \theta = 0 \quad (1)$$

where the dimensionless normalized variables for the temperature θ and radial variable R are

$$\theta = \frac{T - T_\infty}{T_b - T_\infty}, \quad R = \frac{r}{r_2} \quad (2)$$

By means of a variable transformation, it can be shown that equation (1) is a form of Bessel equation [6].

From geometrical considerations, the base of any circular fin coincides with the outer surface of a round tube, meaning that the fin radius, r , never reaches zero. Thereby, the most common boundary conditions for equation (1) are prescribed temperature at the fin base

$$\theta = 1, \quad R = c \quad (3a)$$

and negligible heat loss at the fin tip

$$\frac{d\theta}{dR} = 0, \quad R = 1 \quad (3b)$$

In view of the foregoing, the present fin problem is controlled by two parameters. One is the thermogeometric parameter, $M^2 = h(r_2^3/\delta_1 r_1)/k$, appearing in equation (1). Invariably, M^2 may be also viewed as a modified Biot number. The other parameter, surfacing in equation (3a), is the normalized radii ratio, $0 < c = r_1/r_2 \leq 1$.

The primary specification that thermal design engineers need to address is geared at the level of heat transfer augmentation from a finned tube with respect to the heat transfer from a similar bare tube. Once the dimensionless temperature distribution, $\theta(R)$, is obtained by any solution method, the heat transfer, Q , from an annular fin to a fluid can be computed in two ways: by utilizing the derivative of $\theta(R)$ at the fin base

$$\eta = \frac{Q}{Q_i} = \frac{-2 \left. \frac{d\theta}{dR} \right|_{R=c}}{M^2(1-c^2)} \quad (4)$$

or by employing the integral of $\theta(R)$ over the fin length

$$\eta = \frac{Q}{Q_i} = \frac{-2 \int_c^1 \theta R dR}{(1-c^2)} \quad (5)$$

Note that when $\theta(R)$ is exact, the outcome of equations (4) and (5) is identical, because the heat loss from the fin tip is zero. However, this parity is not always possible when $\theta(R)$ is approximate. Under these circumstances, equation (5) is preferred to equation (4) [6].

Additionally, when designing fins, an important temperature that cannot be overlooked is the tip temperature, i.e., $\theta_t = \theta(1)$ in dimensionless form. As expected, this local temperature must be within tolerable bounds to avoid burns, so that the safety of technical personnel working in plant environments is guaranteed [7].

Exact analytic solution

Equation (1) subject to equation (3) admits an exact analytic solution [4]. The exact dimensionless temperature distribution $\theta(R)$ is:

$$\theta(R) = \sqrt{\frac{R}{c}} \left[\frac{I_{1/3}\left(\frac{2}{3}MR^{3/2}\right)I_{2/3}\left(\frac{2}{3}M\right) - I_{-1/3}\left(\frac{2}{3}MR^{3/2}\right)I_{-2/3}\left(\frac{2}{3}M\right)}{I_{1/3}\left(\frac{2}{3}Mc^{3/2}\right)I_{2/3}\left(\frac{2}{3}M\right) - I_{-1/3}\left(\frac{2}{3}Mc^{3/2}\right)I_{-2/3}\left(\frac{2}{3}M\right)} \right] \tag{6}$$

where $I_\nu(*)$ is the modified Bessel function of first kind and order ν .

Inserting equation (6) into equations (4) or (5) gives way to an exact expression for the fin efficiency. The resulting expression is omitted for brevity but it should be added that its structure is more complicated than equation (6) owing to the appearance of $H_\nu(*)$ together with $I_\nu(*)$.

At this point, we may speculate why the near-optimal circular fin of hyperbolic profile is not presented in textbooks on heat transfer. The answer is fairly simple in view of the difficulties in manipulating Bessel functions to solve equations (1) and (3) analytically. This is further compounded by the laborious evaluation of the awkward temperature distribution in equation (6) and the companion fin efficiency using equations (4) and (5). Recall that the majority of undergraduate students are not proficient in the use of Bessel functions. As we have remarked, these obstacles suffice to justify the search for other avenues for solving equation (1).

Approximate methods

Power series method

The power series method is a robust analytical method for solving linear differential equations with variable coefficients that conform to the general type [8]:

$$y'' + p(x)y' + q(x) = 0 \tag{7}$$

In this equation, the variable coefficients $p(x)$ and $q(x)$ are most likely polynomials represented by power series. Accordingly, a solution of equation (1) is viable in terms of power series of the form

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots = \sum_{m=0}^{\infty} a_m(x - x_0)^m \tag{8}$$

where the constants a_0, a_1, a_2, \dots are the coefficients and x_0 is the center of the power series.

The comparison of equation (1) with equation (7) discloses that the variable coefficients are $p(x) = 0$ and $q(x) = -M^2R$. Consequently, equation (1) is amenable to the power series method and its approximate solution may be readily written as

$$\theta(R) = a_0 + a_1(R - R_0) + a_2(R - R_0)^2 + \dots = \sum_{j=0}^{\infty} a_j(R - R_0)^j \tag{9}$$

The theory of the power series method is presented by Ince [9].

Finite-difference technique

The fin region $c \leq R \leq 1$ is divided into I equal intervals of size $\Delta R = (1 - c)/I$, resulting in $I + 1$ nodes for $i = 0, 1, 2, \dots, I$, as illustrated in Fig. 2. Next, the derivative of second order in equation (1) is represented by the central formulation [10]

$$\left. \frac{d^2\theta}{dR^2} \right|_i = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta R)^2} \quad (10)$$

with a truncation error of order $(\Delta R)^2$. Introduction of equation (10) into equation (1) produces the finite-difference equation

$$\theta_{i+1} - [2 + M^2(\Delta R)^2 R_i] \theta_i + \theta_{i-1} = 0 \quad (11)$$

The preceding equation is an 'equation generator' which furnishes a system of i algebraic equations for the node temperatures θ_i , $i = 1, 2, \dots, I$. The dimensionless thermogeometric parameter, M^2 , and the mesh size, ΔR , arising in equation (11) are quantities known in advance.

Two additional relations are needed to equalize the number of equations in the system of equation (11) to the number of node temperatures, θ_i . The two relations come from the finite-difference representation of the two boundary conditions. One is equation (3a) applied at the first node, $i = 0$ (the fin base), and the other is equation (3b) applied at the last node, $i = I$ (the fin tip). Incorporation of the two boundary conditions in finite-difference form is an easy task because the base temperature is prescribed, simply specifying $\theta_0 = 1$ in equation (11). Also, at the last node, $i = I$, the derivative of first order is represented by a two-point central formulation [10]:

$$\left. \frac{d\theta}{dR} \right|_I = \frac{\theta_{I+1} - \theta_{I-1}}{2\Delta R} \quad (12)$$

with a truncation error of order $(\Delta R)^2$. Application of this formulation sets off the equality $\theta_{I+1} = \theta_{I-1}$ in equation (11).

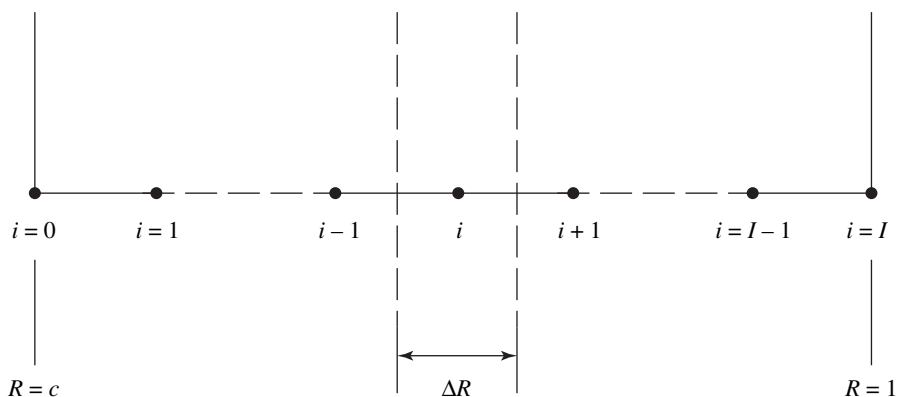


Fig. 2 Nomenclature for the finite-difference technique.

Shooting method

The shooting method is a numerical method that basically consists in a mathematical transformation of a two-point boundary value problem into an equivalent, though incomplete, initial value problem [8]. Let $z = d\theta/dR$, then the second-order differential equation (1) is converted into the following system of two differential equations of first order

$$\frac{d\theta}{dR} = z \quad (13)$$

$$\frac{dz}{dR} = M^2 R \theta \quad (14)$$

Here, the dependent variables are the temperature, θ , and the temperature gradient, z , whereas the independent variable is the radial coordinate, R , in the domain $[c, 1]$.

Evidently, the present situation demands the specification of two boundary conditions at the left extreme, $R = c$: one for θ as given by equation (3a), but the other, for z , is missing. Obviously, the latter needs to be assumed. The test of the assumption is to match the far away boundary condition $d\theta/dR = 0$ at the right extreme $R = 1$ (equation (3b)). When rewritten in terms of z , the boundary condition of equation (3b) simply becomes $z(1) = 0$. In view of the ensuing incompatibility of the boundary conditions, the underlying idea behind the shooting method is to march from the initial point $R = c$, engaging a guessed value of z with the intent of retrieving the far away boundary condition located at the terminal point $R = 1$.

In order to initiate the numerical calculations in the system of equations (13) and (14), we need to guess a value of z at $R = c$, say $z(c) = A$. The solution is then obtained by numerical integrating equations (13) and (14) simultaneously with a fourth-order Runge–Kutta algorithm [10]. A first set of numerical predictions for $z(1)$ and $\theta(1)$ is computed at the end of the interval, and most likely $z(1)$ differs from the required boundary condition $z(1) = 0$. Naturally, we can make a second guess, $z(c) = B$, and perform the numerical integration of the system for the second time. A second set of numerical predications for $z(1)$ and $\theta(1)$ arises at the end of the interval, and again most probably $z(1)$ disagrees with the required boundary condition of $z(1) = 0$.

Instead of continuing with the guessing process, a viable short cut may be implemented upon exploiting the linearity of the original differential equation (1) [10]. Conceptually, the two guessed values, $z(c) = A$ and $z(c) = B$, as well as the two computed values, say $z(1) = D$ and $\theta(1) = E$, are linearly related. As a consequence, these two pairs of numbers, one at the end $R = c$ and the other at the end $R = 1$, can be retained as input data in order to interpolate a third value of $z(0) = C$, to match the far away boundary condition at $z(1) = 0$. This short cut may be easily accomplished with a simple linear interpolation formula [10]. Finally, the third value, $z(c) = C$, can then be used to carry out the numerical integration of the system for the third time. The third try normally should produce the correct or nearly correct recovery of the exact boundary condition $z(1) = 0$. In the unlikely event that this is not the case, the above procedure is repeated until a satisfactory answer is obtained. In the end,

the numerical solutions $z(R)$ and $\theta(R)$ of the system of differential equations (13) and (14) in the domain $c \leq R \leq 1$ should be established with great precision.

We therefore conclude that equations (1) and (3) can always be solved by doing only two computations based on two different guesses for the initial slope. Strangely, this versatile numerical method is never presented in textbooks on heat transfer.

Test problem

Consider a circular fin of hyperbolic profile made from a metallic material of constant thermal conductivity k enveloping a round tube of radius r_1 . Heat is rejected from the fin to a fluid characterized by a fluid temperature T_∞ and a uniform convective coefficient h . As usual, the fin base is maintained at a temperature T_b , and the heat loss through the fin tip may be neglected. A certain fin/fluid assembly is described by a dimensionless thermogeometric parameter $M^2 = 6.572$ and a normalized radii ratio $c = \frac{1}{4}$. Use the power series method, the finite-difference technique and the shooting method to calculate the dimensionless temperature distribution and the fin efficiency. To assess the validity of the three approximate approaches, the results are compared against the analytical temperature distribution of equation (6) and its derived fin efficiency. In this regard, the exact value of $\eta = 0.523$ was taken from the fin efficiency diagram published in [5].

Power series method

Let us choose for convenience $R_0 = 1$ (the fin tip) as the center of the power series in equation (9). We begin the calculations with a five-term series. Introducing the truncated series of equation (9) into equations (1) and (3) and skipping the algebra, the five-term dimensionless temperature distribution as determined by Maple [5] is:

$$\theta(R) = 0.38255 + 1.2571(R-1)^2 + 0.41902(R-1)^3 + 0.68845(R-1)^4 + 0.55076(R-1)^5 \quad (15)$$

Next, repeating the process for a 10-term series, the dimensionless temperature distribution obtained by Maple [5] is:

$$\theta(R) = 0.37329 + 1.2266(R-1)^2 + 0.40888(R-1)^3 + 0.67178(R-1)^4 + 0.53743(R-1)^5 + 0.23674(R-1)^6 + 0.18921(R-1)^7 + 0.09085(R-1)^8 + 0.03888(R-1)^9 + 0.02045(R-1)^{10} \quad (16)$$

Fig. 3 displays the closeness of the temperature distributions for 5 and 10 terms, and shows that both power series handle this complex fin problem reasonably well.

To be on the safe side, the fin efficiency, η , was computed with a 10-term power series. Thereby, equation (16) was introduced into equations (4) and (5) and the results given by the differentiation and integration approaches produced by Maple [5] are $\eta = 0.523$ and 0.522 , respectively. For comparison purposes, the exact fin efficiency is $\eta = 0.523$ [4].

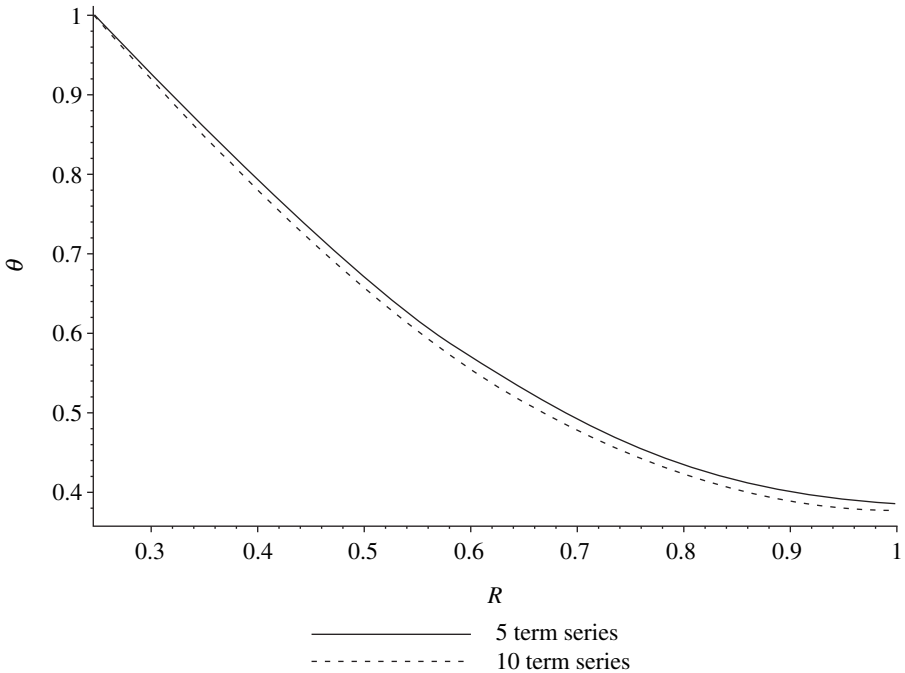


Fig. 3 Comparison of the temperature distributions produced by two power series: one with 5 terms and the other with 10 terms.

Finite-difference technique

As illustrated in Fig. 4, the circular fin of hyperbolic profile is first divided into three equal intervals of size $\Delta R = 1/4$. Accordingly, from the 'equation generator' (11), the three nodal temperatures θ_1 , θ_2 , θ_3 are regulated by the following system of three algebraic equations:

$$\begin{aligned}
 i = 1: 2.206\theta_1 - \theta_2 &= 1 \\
 i = 2: \theta_1 - 2.308\theta_2 + \theta_3 &= 0 \\
 i = 3: 2\theta_2 - 2.411\theta_3 &= 0
 \end{aligned} \tag{17}$$

Clearly, such a small system of algebraic equations can be solved by hand. The computed nodal temperatures, together with the base temperature $\theta_0 = 1$, are listed in Table 1. Since $\Delta R = 1/4$, the tabulated temperature distribution possesses a truncation error of order $1/16$.

Next, the fin efficiencies η may be determined by hand, too, using both equation (4) and equation (5). First, a two-point forward formulation for the derivative of first order in equation (4) gives [10]:

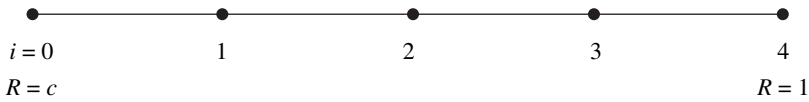


Fig. 4 Location of nodes for a coarse mesh size with intervals $\Delta R = 1/4$.

TABLE 1 Computed nodal temperatures from the finite-difference technique

R	θ
$R_0 = 1/4$	$\theta_0 = 1$
$R_1 = 1/2$	$\theta_1 = 0.653$
$R_2 = 3/4$	$\theta_2 = 0.442$
$R_3 = 1$	$\theta_3 = 0.367$

$$\left. \frac{d\theta}{dR} \right|_{R=c} = \frac{\theta_1 - \theta_0}{\Delta R} \quad (18)$$

where the truncation error is of order ΔR . In addition, a three-point forward formulation for the derivative of first order in equation (4) gives:

$$\left. \frac{d\theta}{dR} \right|_{R=c} = \frac{-3\theta_0 + 4\theta_1 - \theta_2}{2\Delta R} \quad (19)$$

where the truncation error is of order $(\Delta R)^2$ [10]. Substituting the appropriate nodal temperatures in equations (18) and (19) and doing the algebra, the respective fin efficiencies from equation (4) are $\eta = 0.451$ for a two-point approximation and 0.540 for a three-point approximation.

Second, the numerical integration of the integral I in equation (5) can be performed by the trapezoidal rule:

$$I = \frac{\Delta R}{2} (\theta_0 R_0 + 2\theta_1 R_1 + 2\theta_2 R_2 + \theta_3 R_3) \quad (20)$$

whose truncation error is of order ΔR [10]. Moreover, the numerical integration using Simpson's 3/8 rule yields:

$$I = \frac{3\Delta R}{8} (\theta_0 R_0 + 3\theta_1 R_1 + 3\theta_2 R_2 + \theta_3 R_3) \quad (21)$$

whose truncation error is of order $(\Delta R)^2$ [10]. Thus, the fin efficiencies furnished by the trapezoidal rule is $\eta = 0.516$ and by Simpson's 3/8 rule is $\eta = 0.519$.

In general, to assess the validity of the simple finite-difference solution, it is mandatory to compare the numerical results against the analytical solution of equation (6). Otherwise, the mesh needs to be refined and a new set of nodal tempera-

tures computed. We opted here for the former alternative, since we have access to the exact value of $\eta = 0.523$ from the fin efficiency diagram [4]. With the exception of the calculation with the two-point forward formulation for the derivative of first order in equation (4), the accuracy of the three computed efficiencies is high.

If, for other values of the parameters M^2 and c , the coarse mesh with intervals $\Delta R = 1/4$ needs to be halved, this operation gives rise to a new mesh with intervals $\Delta R = 1/8$ and the number of equations in the new system, equation (11), doubles to eight. The numerical calculations of the system with eight equations may be performed with the Gauss elimination method with the symbolic algebra software Maple [5]. From here, the computation of the corresponding fin efficiency is straightforward.

Shooting method

The nature of the mixed boundary conditions in equations (3a) and (3b) suggests that the slope $z = d\theta/dR$ of the temperature curve $\theta(R)$ is always negative in the R domain $[1/4, 1]$. Since $\theta(1/4) = 1$, the largest magnitude of the slope z occurs at the fin base $R = 1/4$; thereafter z decreases monotonically with R and falls to zero value at the fin tip, where $R = 1$. A logical initial guess for $z(1/4)$ may come from equation (19) in conjunction with the finite-difference temperature distribution (Table 1). Doing this, a reasonable guess for the magnitude of $z(1/4)$ is -1.66 . The actual interplay between the step size, h , for the fourth-order Runge–Kutta integration algorithm and the initial slope, $z(1/4)$, was carried out with Maple [5]. Fig. 5a compares the two step sizes $h = 0.1$ and 0.01 and Fig. 5b compares the two step sizes $h = 0.01$ and 0.001 . With regard to the convergence patterns of θ and $Z = d\theta/dR$ and the successful retrieval of the far away slope $z(1) = d\theta(1)/dR = 0$, the conclusion that may be drawn is that the correct initial slope needs to be $z(1/4) = d\theta(1/4)/dR = -1.61$ and the optimal step size must be $h = 0.001$.

Ultimately, inserting the correct initial temperature slope, $z = d\theta(1/4)/dR = -1.61$, into equation (4) delivers the approximate efficiency $\eta = 0.519$. This number compares favorably with the exact efficiency of 0.523 via modified Bessel functions [4].

Conclusions

We have successfully applied the power series method, the finite-difference technique and the shooting method to the Bessel equation governing the temperature variation along a circular fin of hyperbolic profile. The use of Bessel functions was avoided. A comparison test was conducted for a circular fin of hyperbolic profile and with characteristics conducive to a relatively bad efficiency of 0.5 . Under these adverse conditions, the truncated five-term series and the finite-difference approach resulting in a system of four algebraic equations produce results of satisfactory quality. For circular fins of hyperbolic profile holding efficiencies higher than 0.5 , the temperature distributions move up gradually, necessitating the retention of fewer terms in the power series and also smaller systems of algebraic equations. It was also confirmed that the shooting method is more complicated than the power series method and the finite-difference technique, and takes more time. The outcome of this paper may motivate authors of textbooks on heat transfer emphasize the power

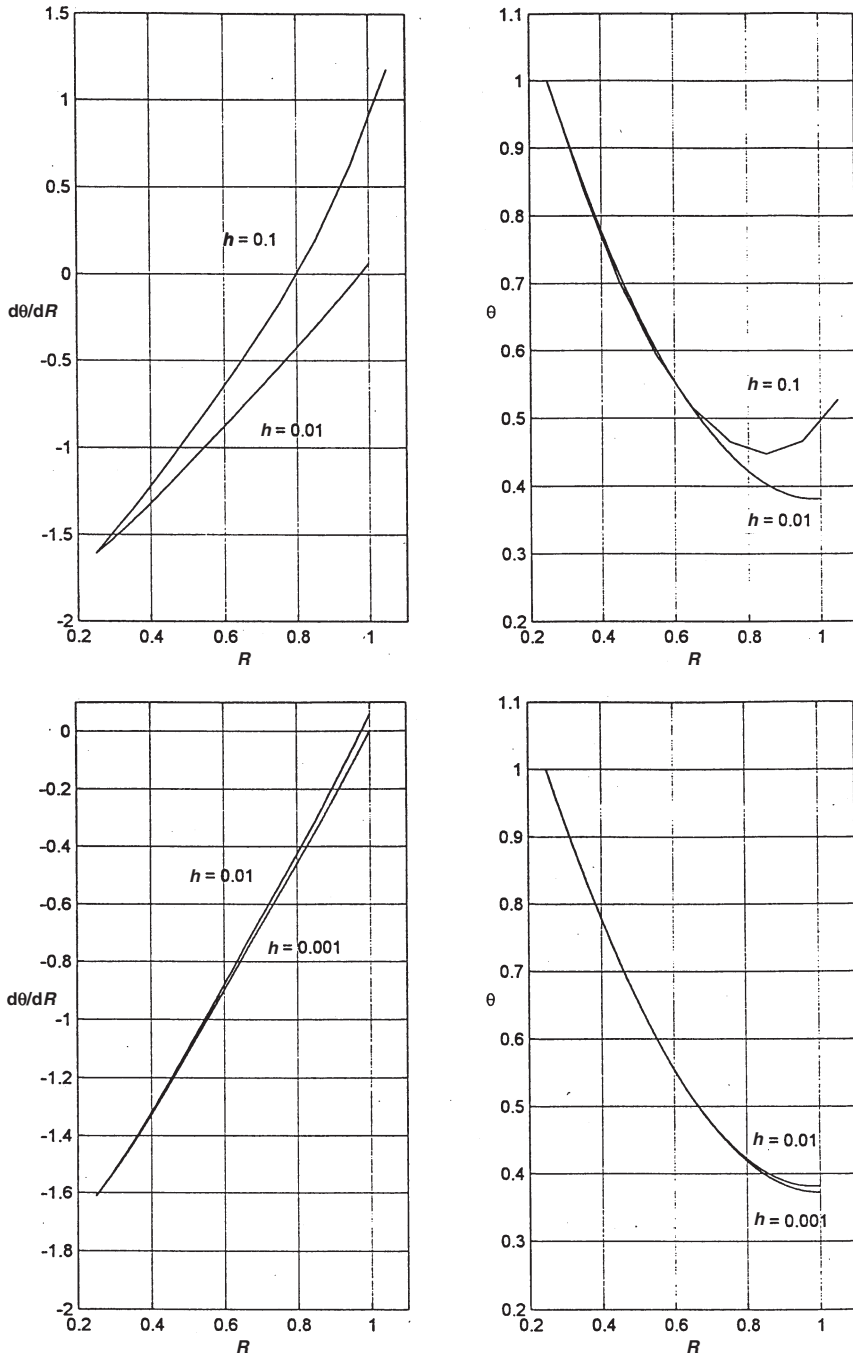


Fig. 5 Convergence patterns exhibited by the temperature distribution and the temperature gradient distribution employing the shooting method.

series method and the finite-difference technique for the analysis of a general class of straight or circular fins of uniform or variable cross-section.

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