
Mathematical modelling of the laminar entrance flow into a circular tube using a Fourier series approximation

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Abstract In this work, the permanent laminar flow in the entrance region of a cylindrical tube is analyzed using a new Fourier series-based approximation technique of similarity for velocity profiles. The similarity method used in this work is well known in the literature on fluid mechanics and can be applied in other engineering education fields. The potential of the original modified Fourier cosine series is compared with the mathematical modelling of the entrance flow. Different physical entrance flow parameters of the interest are obtained through the integral formulation of differential equations for conservation laws. The results of this mathematical approach were found to be compatible with those reported in the literature.

Keywords entrance region; laminar flow; circular tube; similarity method; Fourier series

Introduction

Exact solutions to problems with differential formulations involving the motion of fluids are very difficult, or even impossible, to obtain, even when the geometry is simple and the fluid's physical properties are constant. Numerical solutions are usually good options but, when an analytical description is required, approximate methods of formulation and solution are often useful.

The partial differential formulation of the circular entrance flow contains the non-linear terms of the convective acceleration caused by the boundary layer velocity developing on the tube's internal wall. Although this classical problem of fluid mechanics is well characterized in the literature, recent studies have focused on the entrance region involved in the forced transport of special non-Newtonian fluids [1], blood [2] and liquid mixtures [3].

The similarity method is a technique which reduces a partial differential equation in two independent variables to an ordinary differential equation involving only a single variable. Sometimes called 'combination variables' or associated in the literature with the integral method, similarity analysis is applicable to certain problems in which the characteristic lengths are determined by rate processes rather than by the geometric or physical dimensions. Such problems generally involve regions which are regarded as being semi-infinite. Examples include transient diffusion or conduction and various steady, two-dimensional flow problems involving velocity, temperature, and/or concentration boundary layers [4].

With this similarity method it is necessary that the field variable of velocity has profiles which are identical in shape for all positions or times, differing only by the scale over which the variations occur and described by a variable cinematic of similarity. Von Kármán [5] and Pohlhausen [6] developed this approximate method, which has been used with considerable success for the analysis of boundary layer flows. The method is based on an integral formulation of the problem and its result, the calculated flow field, usually satisfies the equations of continuity and momentum. The boundary conditions used for the flow over a solid surface are increased to polynomial functions as an approach for the velocity profiles in the laminar boundary layer.

The entrance flow into a circular tube is a typical problem of this nature; the integral approximate method was used by Schiller [7] to model it. In this work, the permanent laminar flow in the entrance region of a cylindrical tube is analyzed using a new Fourier series-based approximation technique for velocity profiles. The potential of the original modified Fourier cosine series is compared with the mathematical modelling of the entrance flow. The main motivation of this approach is to obtain in series the field of velocity, to date described in this form only for the bi-dimensional laminar flow in a rectangular duct [8].

Formulation of the problem

Fig. 1 illustrates a steady entrance flow into a cylindrical pipe. The fluid flows from a large reservoir into a long straight pipe equipped with a bell-shaped mouth to prevent perturbation. The flow is permanent, incompressible, and laminar, and the fluid is Newtonian.

The radial component of the convective acceleration is much smaller than the axial component [9]; however, $u_x = u_x(x, r)$ in the boundary layer of the entrance region; $u_x = u_x(x)$ in the potential core of the entrance region; and $u_x = u_x(r)$ begins at the end of the potential core.

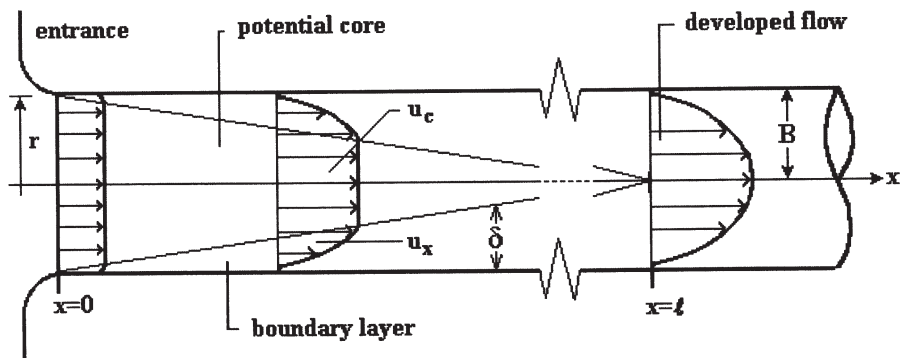


Fig. 1 Entrance flow into a cylindrical pipe.

The velocity at the mouth of the pipe is uniform. A boundary layer of thickness δ increases from zero at the mouth to the full radius (B) of the pipe at the end of the entrance region, where $x = \ell$. For any x value between zero and ℓ , the velocity profile displays a gradient across the boundary layer thickness and a constant value across the potential core. The potential core disappears at the end of the entrance region; from then on, the flow is established and the velocity profile takes on the classical parabolic form. The conditions at the end of the entrance zone are:

$$u_x = U \quad \text{at } x = 0 \quad (1)$$

$$u_x = 2U[1 - (r/B)^2] \quad \text{at } x = \ell \quad (2)$$

The differential formulation for the problem is given by the equations of continuity, momentum and energy in cylindrical coordinates, which, in terms of dimensionless parameters, are written as follows:

$$\frac{\partial U_x}{\partial X} + \frac{1}{R} \frac{\partial}{\partial R}(RU_r) = 0 \quad (3)$$

$$U_x \frac{\partial U_x}{\partial X} + U_r \frac{\partial U_x}{\partial R} = -\frac{dP}{dX} + \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial U_x}{\partial R} \right) \quad (4)$$

$$\frac{1}{2} \frac{\partial U_x^3}{\partial X} + \frac{1}{2} \frac{\partial (U_r U_x^2)}{\partial R} = -U_x \frac{dP}{dX} + \frac{1}{R} \frac{\partial}{\partial R} \left(U_x R \frac{\partial U_x}{\partial R} \right) \quad (5)$$

in which $X = vx/B^2U$, $L = v\ell/B^2U$, $R = r/B$, $U_x = u_x/U$, $U_r = u_r/B/v$, and $P = p'/\rho U^2$.

These equations are used to obtain the integral formulation by means of their integration over the cross-sectional area of the flow. A detailed procedure is given by Campbell and Slaterry [10], and by Fargie and Martin [11]. The final results are:

$$\int_0^1 U_x R dR = \frac{1}{2} \quad (6)$$

$$-\frac{1}{2} \frac{dP}{dX} = \frac{d}{dX} \int_0^1 U_x^2 R dR - \frac{\partial U_x}{\partial R} \Big|_{R=1} \quad (7)$$

$$-\frac{dP}{dX} = \frac{d}{dX} \int_0^1 U_x^3 R dR + 2 \int_0^1 \left(\frac{\partial U_x}{\partial R} \right)^2 R dR \quad (8)$$

To obtain a solution, an approximate functional specification, $U_x = U_x(X, R)$, is needed to describe the flow velocity profile as x varies from zero to L , or, in terms of dimensionless variables, as X varies from zero to $L = v\ell/B^2U$.

Specification of the velocity profile based on a Fourier series

The velocity profiles at the mouth and at the end of the entrance region are given by equations 1 and 2, or, with the dimensionless variables, by:

$$U_x = 1 \quad \text{at } X = 0 \quad (9)$$

$$U_x = 2(1 - R^2) \quad \text{at } X = L \quad (10)$$

These two functions, which can be represented by Fourier cosine series over the interval (0,1) of the variable R , are, respectively:

$$U_x(0, R) = 1 = g(R) = \sum_0^\infty \left(\frac{2}{\lambda_n}\right) \cos(\lambda_n R) \tag{11}$$

$$U_x(1, R) = 2(1 - R^2) = h(R) = \sum_0^\infty \left(\frac{2}{\lambda_n}\right)^3 \cos(\lambda_n R) \tag{12}$$

in which $\lambda_n = (2n + 1)\pi/2$.

It is a remarkable fact, and considered potentially very useful, that the only difference between the two series is the exponent of the first factor, which is 1 for the flat profile and 3 for the parabolic profile.

A single parameter, say $k = k(X)$, could be substituted for both and act as a shape factor, producing the flat profile when it is equal to 1 and the parabolic profile when it is equal to 3, and, hopefully, a physically adequate transition profile for $0 < X < 1$.

Therefore, the tentative velocity profile for the entrance region becomes:

$$U_x(X, R) = \sum_0^\infty \left(\frac{2}{\lambda_n}\right)^k \cos(\lambda_n R) \tag{13}$$

Obviously, the next question is whether or not this profile would actually describe the physical velocity transition over the entrance region. Fig. 2 aims to help answer this question.

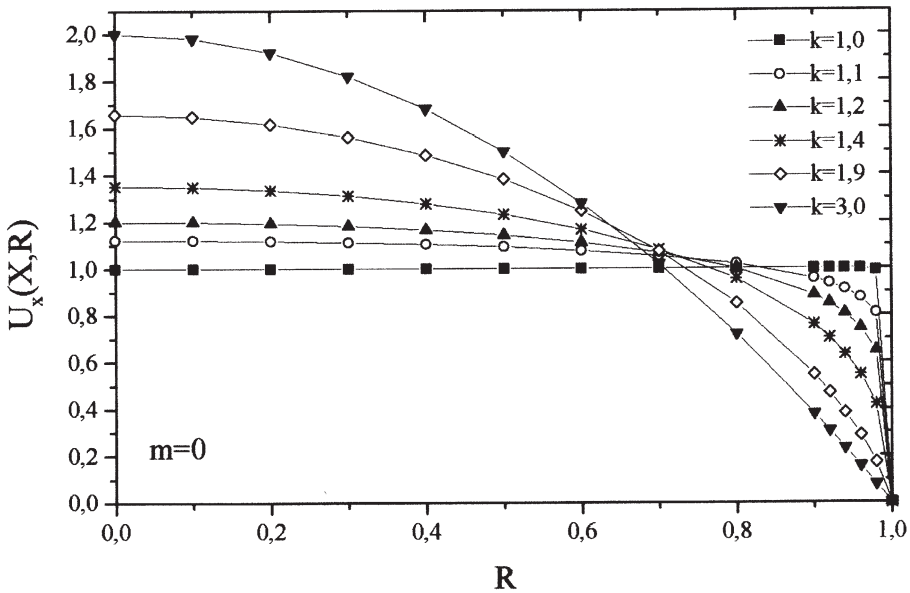


Fig. 2 Parameterized Fourier series representation for the velocity profile at the entrance region of a cylindrical duct.

It is apparent that the general features of the flow are described, but the changes are initially very fast. A small increase in k from 1.0 to 1.2, for instance, produces a velocity profile that is already disturbed all the way down to the centerline of the flow. The flatness of the velocity profile, which should have been preserved at this point of the potential core, has disappeared almost completely. The description can be improved, though, through the introduction of changes in the way the variable R is weighted as k varies from 1 to 3. The following proposal is consistent with the original one when $m = 0$, and produces changes in the right direction with increasingly higher values of m :

$$U_x(X, R) = \sum_0^{\infty} \left(\frac{2}{\lambda_n} \right)^k \cos(\lambda_n R^{[1+m(3-k)]}) \quad (14)$$

The new series are not actual Fourier series, for $m \neq 0$, except when $k = 3$ (parabolic profile). They will be referred to as modified Fourier series, or MFS, in the text. In equation 14, the numeric variation of the exponent $(1 + m(3 - k))$ alters the results of the velocity profile. This change, as a function of m , produces a better description of the velocity profile core, as illustrated in Fig. 3, which shows the velocity profiles for two values of m : $m = 1$ (Fig. 3a) and $m = 3$ (Fig. 3b), with a better core definition for $m = 3$.

For the velocity field in Fig. 3 to represent the flow in the entrance region, $k(X)$ must be specified. This is done by transferring the pressure gradient from equation 7 to equation 8, resulting in:

$$\frac{d}{dX} \left[2 \int_0^1 U_x^2 R dR - \int_0^1 U_x^3 R dR \right] = 2 \frac{\partial U_x}{\partial R} \Big|_{R=1} + 2 \int_0^1 \left(\frac{\partial U_x}{\partial R} \right)^2 R dR \quad (15)$$

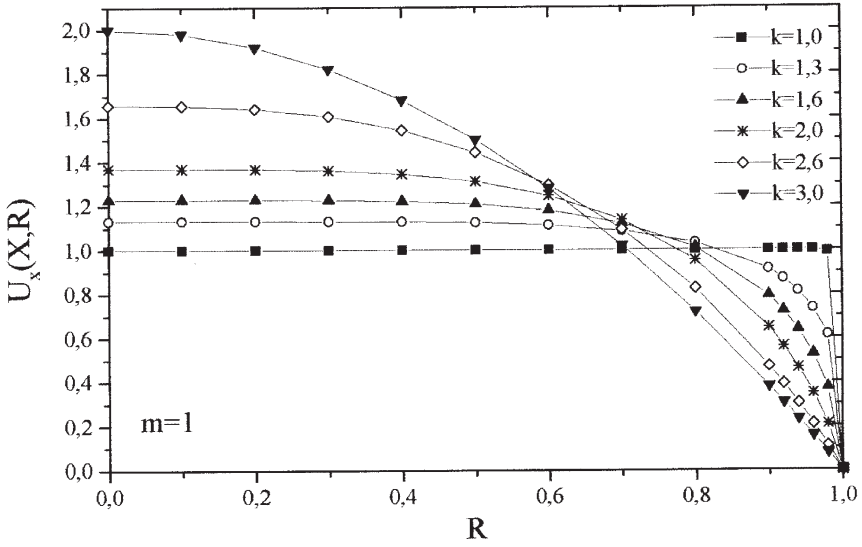
Equation 15 can thus be written as:

$$dX = \left[\frac{4 \int_0^1 U_x \frac{\partial U_x}{\partial k} k R dR - 3 \int_0^1 U_x^2 \frac{dU_x}{dk} R dR}{2 \frac{\partial U_x}{\partial R} \Big|_{R=1} + 2 \int_0^1 \left(\frac{\partial U_x}{\partial R} \right)^2 R dR} \right] dk \quad (16)$$

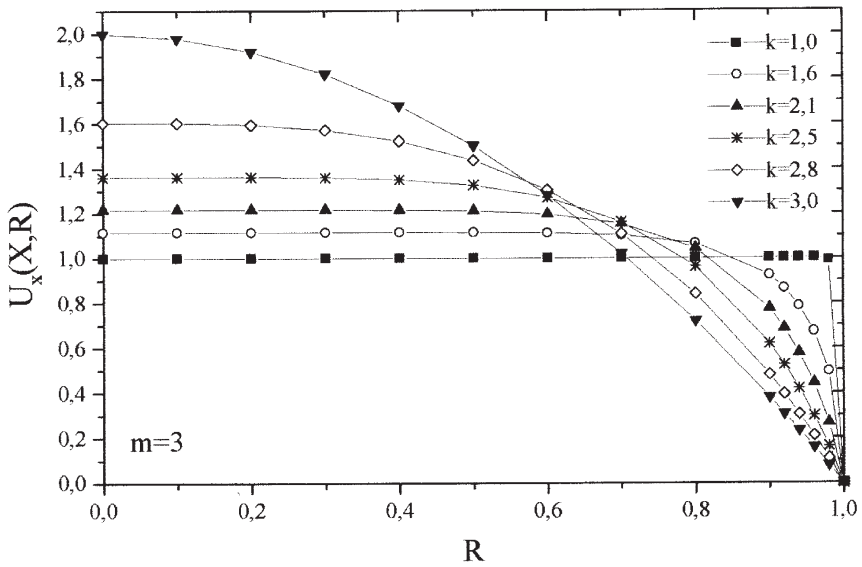
and integrated to produce:

$$X = \int_1^k \left[\frac{4 \int_0^1 U_x \frac{dU_x}{dk} R dR - 3 \int_0^1 U_x^2 \frac{dU_x}{dk} R dR}{2 \frac{dU_x}{dR} \Big|_{R=1} + 2 \int_0^1 \left(\frac{dU_x}{dR} \right)^2 R dR} \right] dk \quad (17)$$

By transferring the velocity profile given by equations 14–16, one obtains an expression for X , $X = \int_1^k (\text{function of } k) dk$, as a definite integral involving k . This operation is performed numerically, producing $k(X)$ for any m value, thus specifying the velocity values anywhere at the entrance.



(a)



(b)

Fig. 3 Modified Fourier series (MFS) representation for the velocity profile in the entrance region. (a) MFS for $m = 1$; (b) MFS for $m = 3$.

Results and comparison with the literature

Flow velocities along the tube axis ($R = 0.0$) for different values of m are shown in Fig. 4 and compared with those of other models. The solution for $m = 0$ displays an abrupt increase in velocity close to the entrance flow, which is very different behavior to that observed for $m = 1, 2, 3$ and 4.

Fargie and Martin [11] and Al-Nassry and Unny [12] used polynomial solutions for the approximate velocity profiles, while Sparrow *et al.* [9] presented an exact solution for the linearized momentum equation. The solution found in this work for $m = 4$ is very close to the results of Al-Nassry and Unny [12].

Fig. 5 displays the results for velocities in the radial position ($R = 0,4$) for $m = 1$ and $m = 4$, and the results based on a variational analysis by Tao and Gupta [13], experiments by Restoko (see [14]) and a numerical investigation of finite elements by Comini *et al.* [14].

Fig. 6 illustrates the results given by this model for the additional pressure drop, $D(X)$, which corresponds to the transition from the excess pressure drop occurring in the entrance region to the pressure drop of the established flow. $D(X)$ is given by:

$$D(X) = \frac{\Delta P}{\rho \cdot u^2/2} - \frac{64}{Re} \cdot (x/2B)$$

The results for $m = 4$ are in good agreement with the theoretical approaches of Sparrow *et al.* [9] and Tao and Gupta [13], and the experimental results of McComas and Eckert [15] and Liu (see Shah and London [8]).

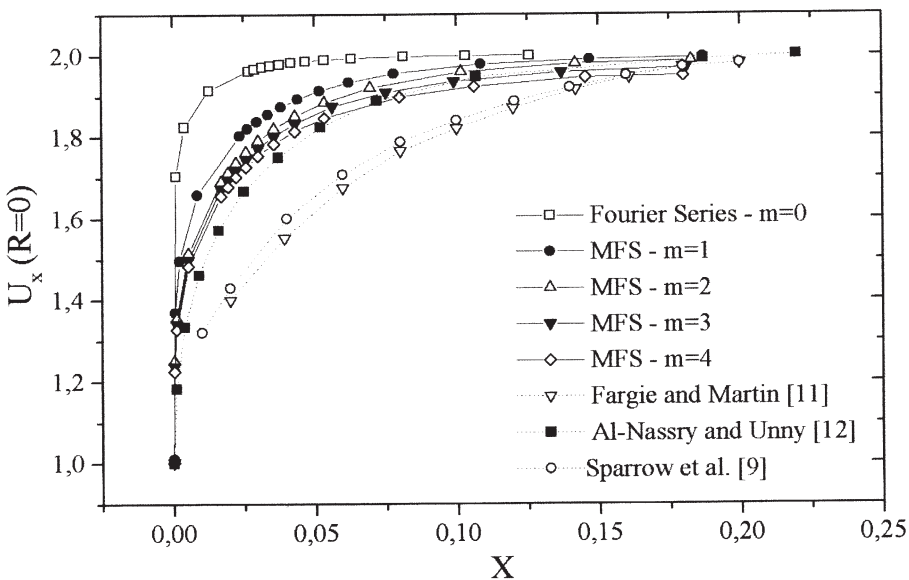


Fig. 4 Axial velocity in the radial position, $R = 0.0$. Comparison with the literature.

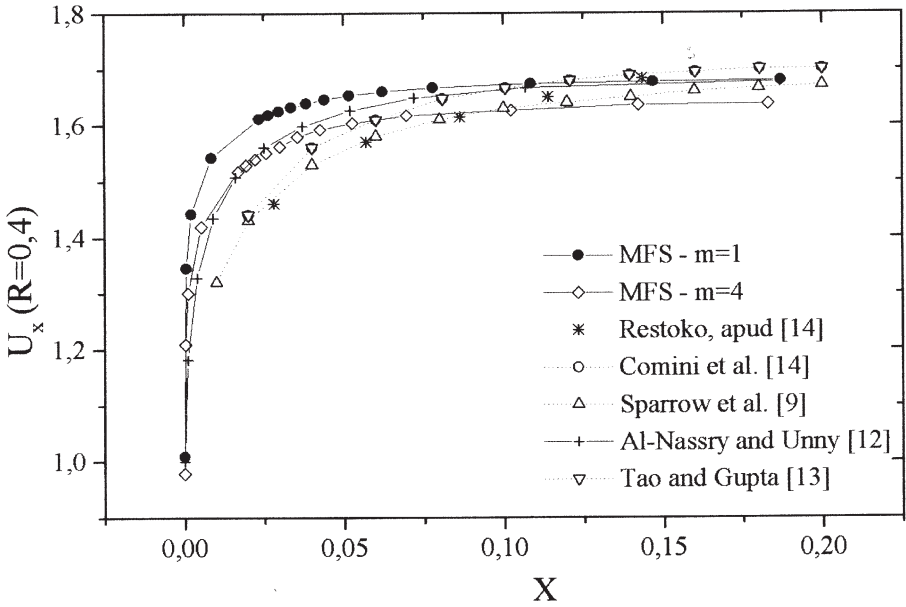


Fig. 5 Axial velocity in the radial position, $R = 0.4$. Comparison with the literature.

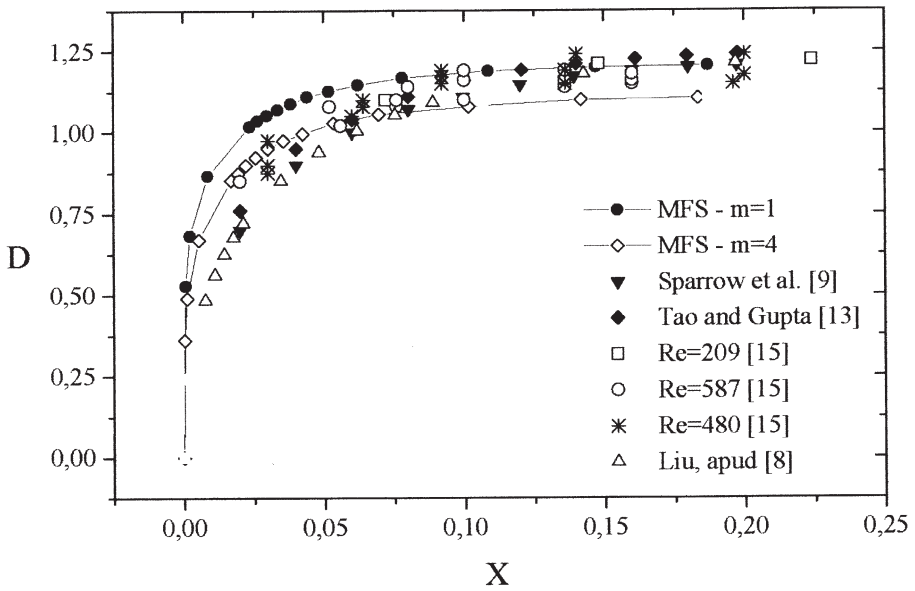


Fig. 6 Additional pressure drop (D) as a function of the axial position. Comparison with the literature.

In comparison with other approaches in the literature, this model has a cinematic variable of similarity (variable m) whose numeric variation can produce different profiles of velocity.

Conclusions

Based on the results and analysis of this mathematical approach to describe the entrance flow, the following conclusions can be drawn. First, the original form the Fourier cosine series for the specification of the velocity profile (equation 13 or equation 14, with $m = 0$) is not in good agreement with the results of other investigators; hence, this series function does not model the entrance flow in a circular tube adequately. Second, the modified Fourier series (MFS) (equation 14) proposed for modelling the entrance flow is consistent and satisfactory for values of m (cinematic variable of similarity) greater than 0.

References

- [1] J. Liang, 'Determination of the entry region length of viscoelastic fluid flow in a channel', *Chemical Engineering Science*, **53**(17) (1998), 3185–3187.
- [2] T. Shipkowitz, V. G. J. Rodgers, L. J. Frazin and K. B. Chandran, 'Numerical study on the effect of steady axial flow development in the human aorta on local shear stresses in abdominal aortic branches', *Journal of Biomech.*, **31**(11) (1998), 995–1007.
- [3] A. Beretta, P. Ferrari, L. Galbiati and P. A. Andreini, 'Horizontal oil–water flow in small diameter tubes. Pressure drop', *International Communications in Heat and Mass Transfer*, **24**(2) (1997), 231–239.
- [4] W. M. Deen, *Analysis of Transport Phenomena* (Oxford University Press, New York, 1998).
- [5] T. Von Kármán, 'Über laminare and turbulente Reibung', *ZAMM*, **1** (1921), 244–247.
- [6] E. Pohlhausen, 'Der Wärmeaustausch zwischen festen Körpern und Flüssigkeiten mit kleiner Reibung und kleiner Wärmeleitung', *ZAMM*, **1** (1921), 121.
- [7] L. Schiller, 'Die Entwicklung der laminaren geschwindigkeitsverteilung und ihre bedeutung für zahigkeitsmessungen', *ZAMM*, **2** (1922), 96–106.
- [8] R. K. Shah and A. L. London, *Laminar Flow Forced Convection in Ducts* (Academic Press, New York, 1978).
- [9] E. M. Sparrow, S. J. Lin and T. S. Lundgren, 'Flow development in the hydrodynamics entrance region of tubes and ducts', *Journal of Physics Fluids*, **7** (1964), 338–347.
- [10] W. D. Campbell and J. C. Slaterry, 'Flow in the entrance of a tube', *Journal of Basic Engineering*, **85** (1963), 41–46.
- [11] D. Fargie and B. W. Martin, 'Developing laminar flow in a pipe of circular cross-section', *Proceedings of the Royal Society of London*, **A321** (1971), 461–476.
- [12] S. A. Al-Nassry and T. Unny, 'Developing laminar flow in the inlet length of a smooth pipe', *Applied Science Research*, **A36** (1980), 313–332.
- [13] L. N. Tao and S. D. Gupta, 'Variational analysis of the flow development in the entrance region of circular tubes and parallel-plate channels', *Proceedings of the 13th Midwestern Mechanics Conference. Developments in Mechanics*, **7** (1964), 247–263.
- [14] G. Comini, S. Giudici and M. Strada, 'Finite element analysis of laminar flow in the entrance region of ducts', *International Journal for Numerical Methods in Engineering*, **15** (1980), 507–517.
- [15] S. T. McComas and E. R. G. Eckert, 'Coefficient K for the pressure drop caused by laminar flow through the inlet section of a tube', *Transactions ASME, Journal Applied Mechanics*, **32** (1965), 765–770.