
Mathematica software in engineering mathematics classes

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Abstract In this paper we show the advantages of using Mathematica software in engineering mathematics classes through the study of an example problem concerning heat conduction in a slab. Firstly the problem is solved from the point of view of a parabolic model of heat conduction, and secondly from the viewpoint of a hyperbolic model.

Keywords Mathematica; mathematics classrooms; hyperbolic heat equation

Introduction

The use of modern and powerful software is a highly useful pedagogical tool in mathematics classes for engineers. Software that can readily produce graphical representations allows students to visualize quickly and easily the information contained in complex formulae obtained as a solution to problems based on partial differential equations. As a consequence of this fact, more advanced engineering topics can be included in classes.

In this paper we use Mathematica to present an example that both illustrates the use of pedagogical tools and shows how high-level problems may be introduced in mathematics engineering classes. To achieve these objectives we study an example of a standard mathematical problem based on heat conduction in a semi-infinite slab. The problem is solved from two different points of view: parabolic and hyperbolic models of heat conduction. With Mathematica we analyze and discuss both solutions obtained.

Heat conduction problem

The problem is based on the heat conduction in a semi-infinite slab whose temperature initially is u_1 and from the initial time ($t = 0$) its left extreme is at constant temperature u_2 . The boundary and initial conditions of the problem are as follows:

$$\forall x > 0 \quad U(x, 0) = u_1 \quad (1)$$

$$\forall t > 0 \quad U(0, t) = u_2 \quad (2)$$

$$\forall t > 0 \quad \lim_{x \rightarrow \infty} U(x, t) = u_1 \quad (3)$$

where $U(x, t)$ is the temperature, x is the spatial variable and t is time.

The governing equation that we are going to use is the heat conduction equation. The expression of this equation depends on the model (parabolic or hyperbolic) used to solve the problem.

Parabolic temperature profile

The parabolic heat conduction equation in one-dimensional solids is:

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} \quad \forall x > 0, \quad t > 0 \quad (4)$$

where α is the thermal diffusivity of the medium (see for example [1]).

The resolution of the problem begins by taking a Laplace transform with respect to t in equation 4. Then, applying conditions 1, 2 and 3, and using the tables for an inverse Laplace transform found in [2], we obtain the parabolic solution of the problem:

$$U(x, t) = u_1 + (u_2 - u_1) \operatorname{Erfc}\left(\frac{x}{2\sqrt{t\alpha}}\right) \quad (5)$$

where $\operatorname{Erfc}(\mu)$ is the complementary error function:

$$\forall \mu \geq 0 \quad \operatorname{Erfc}(\mu) = \frac{2}{\sqrt{\pi}} \int_{\mu}^{\infty} e^{-v^2} dv \quad (6)$$

The parabolic solution (equation 5) shows that for all $x > 0$ and for all $t > 0$ the temperature is always bigger than the initial value, since the complementary error function is always bigger than zero. If initially all the points of the semi-infinite slab are at temperature u_1 and an increase in temperature is due to a contribution of heat, then we can conclude that the parabolic model predicts an infinite speed of heat conduction. Although the parabolic model implies this unacceptable conclusion, this equation continues to be used, since in ordinary engineering applications the equation results agree with the experience.

The important point to note now is the way in which Mathematica software may be used to study the solution obtained. With this software we obtain numerical values and graphical representations of the solution for a specific material. We have chosen aluminium for our specific computations.

Firstly, we have to introduce the necessary data. The diffusivity of aluminium is $\alpha = 85.9 \times 10^{-6} \text{ m}^2/\text{s}$ [1] and the thermal relaxation time is $\tau = 10^{-13} \text{ s}$ [3]. We suppose that $u_1 = 25^\circ \text{ C}$ and $u_2 = 50^\circ \text{ C}$. Secondly, we write the expression which we want to study.

Fig. 1 shows how the data and parabolic temperature profile are entered in Mathematica. The last line shown in Fig. 1 is the instruction to obtain a graphical representation of the parabolic solution over a specific spatial and temporal interval.

One such is shown in Fig. 2, which plots the temperature evolution in the slab from the initial temperature 25° C to 50° C .

An important point, from pedagogical viewpoint, is that Fig. 2 shows graphically the infinite speed of heat conduction (the temperature curve is always above the straight line $U = 25$), a fact which was previously observed from the purely mathematical perspective. However, in Fig. 2 we cannot observe the quantitative difference in temperature. To remedy this fact, in Fig. 3 we obtain precise values of temperatures.

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Mathematica 4 - [paper.nb]
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paper.nb

In[1]:= diffusivity := 85.9 10^-6
In[2]:= relaxation := 10^-13
In[3]:= u1 := 0
In[4]:= u2 := 100
In[5]:= parabolic[x_, t_] := u1 + (u2 - u1) Erfc[x / (2 Sqrt[t diffusivity])]
In[6]:= Plot[parabolic[x, 10^-5], {x, 0, 10^-3}]

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Fig. 1 Data entry in Mathematica.

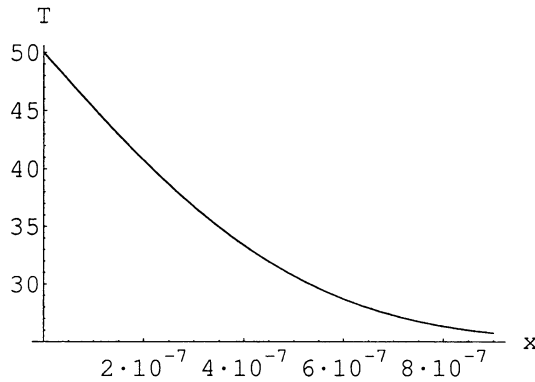


Fig. 2 Plot produced by Mathematica showing temperature as a function of position: parabolic solution.

With Mathematica we can obtain three-dimensional plots too. Fig. 4 shows a three-dimensional graph of the parabolic solution. This lets the student observe the spatial and the temporal temperature evolution at the same time.

Hyperbolic temperature profile

The development of technology has led to new physical situations in which great amounts of heat are applied to materials in very short times. In these situations there are relevant differences between the parabolic predicted results and the actual

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paper.nb *

In[80]:= N[parabolic[9 10-7 , 10-9]]
Out[80]= 25.7476

In[58]:= N[parabolic[10-6 , 10-9]]
Out[58]= 25.396

In[81]:= N[parabolic[1.5 10-6 , 10-9]]
Out[81]= 25.0074

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Fig. 3 Numerical output from Mathematica.

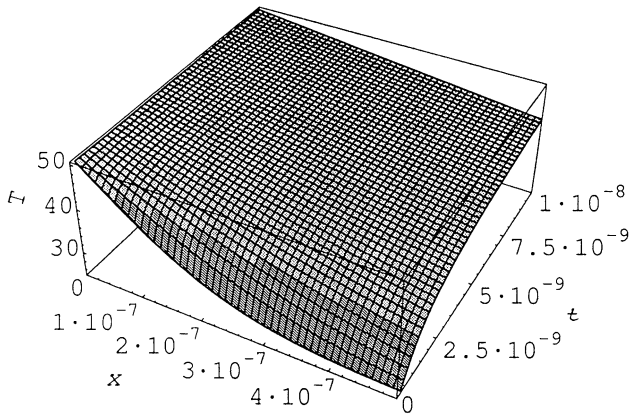


Fig. 4 Three-dimensional plot produced by Mathematica of the change in temperature (T) by position (x) and time (t): parabolic solution.

(i.e. measured) results. This highlighted the need for a new heat conduction model, and the hyperbolic model of heat conduction [e.g. 4] meets that need.

The hyperbolic heat conduction equation turns out to be:

$$\tau \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} \quad \forall x > 0, \quad t > 0 \quad (7)$$

where τ is the thermal relaxation time, which represents the time lag for the beginning of heat flux after a gradient temperature has been imposed.

To solve the problem from the hyperbolic viewpoint, we need to use conditions 1, 2 and 3, and an extra condition due to the presence of second partial derivatives with respect to time in equation 7. The physical meaning of the new condition is the absence of flux in the initial state of the slab:

$$\forall x > 0 \quad \frac{\partial U}{\partial t}(x, 0) = 0 \tag{8}$$

The hyperbolic problem resolution is based on Laplace transforms too. To make a Laplace inverse transform, we use the table shown in [1]. In this way we obtain the following hyperbolic solution:

$$U(x, t) = u_1 + (u_2 - u_1)H\left(t - \frac{x}{v}\right) \left[e^{-\frac{x}{2\sqrt{t\alpha}}} + \frac{xv}{4\alpha t} \int_{\frac{x}{v}}^t e^{-\frac{u}{2\tau}} \frac{I_1\left(\sqrt{\left(\frac{u}{2\tau}\right)^2 - \frac{x^2}{4\tau\alpha}}\right)}{\sqrt{\left(\frac{u}{2\tau}\right)^2 - \frac{x^2}{4\tau\alpha}}} du \right] \tag{9}$$

where $v = \frac{\sqrt{\alpha}}{\sqrt{\tau}}$ is the speed of heat conduction, $I_1(u)$ is the modified Bessel function of the first kind and order 1, and $H(u)$ is the Heaviside function:

$$\forall u \in \mathbb{R}; \quad H(u) = \begin{cases} 1 & u \geq 0 \\ 0 & u < 0 \end{cases}$$

The theoretical deduction of equation 7 is based on a finite speed of heat conduction, v . The presence of the Heaviside function clearly reflects this fact: when $t \leq \frac{x}{v}$, the Heaviside function is zero and the slab is at its initial temperature, but

when $t > \frac{x}{v}$, the Heaviside function is 1 and the temperature of the slab has changed.

Fig. 5 shows the hyperbolic temperature evolution. In this case we see that the temperature of the slab goes from the initial temperature to 50°C as in the parabolic model, but the way in which it does so is very different. In the case of hyperbolic model, for every time t the graphic is divided in two zones: a zone which is at the initial temperature, since heat has not arrived; and a zone where heat has arrived and which is therefore at a higher temperature than initially. This fact is due to the effect of the Heaviside function.

Fig. 5 shows that Mathematica quickly gives a plot that shows visually the hyperbolic pattern of heat conduction. We think that students will appreciate this plot for its high didactic content.

Fig. 6 is a three-dimensional graph of hyperbolic temperature. It shows the spatial and temporal evolution of temperature at the same time, and we can observe better the effect of the Heaviside function.

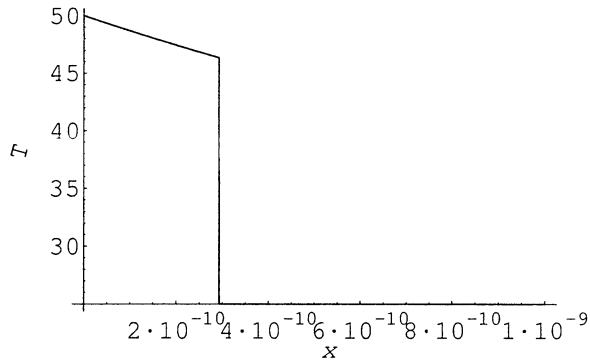


Fig. 5 Plot produced by Mathematica showing temperature as a function of position: hyperbolic solution.

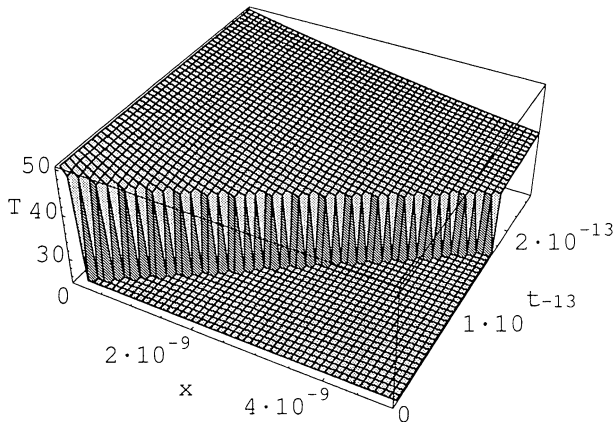


Fig. 6 Three-dimensional plot produced by Mathematica of the change in temperature (T) by position (x) and time (t): hyperbolic solution.

From all the parabolic and hyperbolic graphics, we can conclude that, over short time frames, there are relevant differences between the parabolic and hyperbolic models. For this reason, in processes in which short times are important (pulsed-laser processing of metals and semiconductors, film applications, laser surgery, nuclear engineering, etc.) the parabolic model produces errors and it is necessary to use the hyperbolic equation.

Conclusions

In this paper we have presented an example of the use of Mathematica software in engineering mathematics classes. We have shown the solution of a heat conduction

problem from the point of view of a parabolic and a hyperbolic model of heat conduction and their differences through Mathematica.

We have shown the pedagogical improvement of teachers' work in engineering mathematics classes, since students can interpret solutions graphically and numerically. Through the solution representation students can understand the role played by non-trivial functions (in this case the Heaviside function).

Also, with Mathematica students can relate the theoretical problems solved in class to real problems. The heat conduction problem studied in this paper is a standard one in many engineering mathematics courses.

Acknowledgement

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