On various eigenvalue problem formulations for viscously damped linear mechanical systems

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Abstract The state-space method is frequently used to obtain the eigenvalues of a viscously damped linear mechanical system. Differences in the definition of the state vector and auxiliary matrices found in the literature lead to differences in the formulation of the eigenvalue problems and this in turn can cause difficulties for students on mechanical vibration courses. In this study, various eigenvalue problem formulations in different textbooks have been examined, relationships between them have been established and results have been applied to a numerical example of a system with two degrees of freedom.

Keywords mechanical systems; viscous damping; state-space method; eigenvalue problems

Notation

$\mathbf{0}$ $n \times n$ zero matrix
$\mathbf{C}$ damping matrix
$\mathbf{I}$ $n \times n$ unit matrix
$\mathbf{K}$ stiffness matrix
$\mathbf{M}$ mass matrix
n degrees of freedom of the mechanical system
$\boldsymbol{q}$ vector of generalized coordinates
$\dot{\boldsymbol{q}}$ vector of generalized velocities
$\tilde{\boldsymbol{q}}$ eigenvector of the system in the physical space
$\boldsymbol{x}$ state vector
$\tilde{\boldsymbol{x}}$ eigenvector of the system in the state space
λ eigenvalue of the system

Introduction

In order to obtain the general solution for vibrations of a linear mechanical system which is viscously damped, it is first necessary to determine the eigenvalues of the system. The method which is generally followed for this purpose is to apply the state-space method [1–8]. However, the differences seen in the definitions in the literature of either the state vector or the auxiliary matrices lead to different standard eigenvalue or generalized eigenvalue problem formulations. Due to different formulations in various textbooks, students on courses on mechanical vibrations can have difficulty understanding the relationships between these formulations. Therefore, in the present paper, eigenvalue problem formulations in different textbooks are examined, written in the same notation and compared, and relationships between them are outlined. Following this, a numerical example is given, for a system with two degrees of freedom.

Theory

As is known, the motion of a viscously damped linear mechanical system with n degrees of freedom is governed in the physical space by the following second-order matrix differential equation:

$$\mathbf{M}\ddot{\boldsymbol{q}}(t)+\mathbf{C}\dot{\boldsymbol{q}}(t)+\mathbf{K}\boldsymbol{q}(t)=\mathbf{0} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $(n \times n)$ mass, damping and stiffness matrices, respectively, and $\boldsymbol{q}(t)$ is the $(n \times 1)$ vector of generalized coordinates.

In this study, eigenvalue problem formulations in some textbooks on vibrations and dynamic systems are summarized using the same notation as far as possible.

Eigenvalue problem formulation from Müller and Schiehlen [1], Shabana [2] and Newland [3]

Let the state vector $\boldsymbol{x}(t)$ be defined as the $(2n \times 1)$ vector:

$$\boldsymbol{x}=\begin{bmatrix}\boldsymbol{q}\\ \cdots\\ \dot{\boldsymbol{q}}\end{bmatrix} \tag{2}$$

i.e., the state vector is composed of the generalized coordinate vector $\boldsymbol{q}$ and the generalized velocity vector $\dot{\boldsymbol{q}}$. The second-order differential equation 1 can equivalently be written in the so-called state-space form:

$$\dot{\boldsymbol{x}}(t)=\mathbf{A}\boldsymbol{x}(t) \tag{3}$$

where the $(2n \times 2n)$ system matrix $\mathbf{A}$ is defined as:

$$\mathbf{A}=\begin{bmatrix}\mathbf{0} & \vdots & \mathbf{I}\\ \cdots & \vdots & \cdots\\ -\mathbf{M}^{-1}\mathbf{K} & \vdots & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix} \tag{4}$$

$\mathbf{0}$ and $\mathbf{I}$ being the $(n \times n)$ zero matrix and unit matrix, respectively. It is obvious that the matrix differential equation 3 in the state space is a first-order differential equation.

Let it be assumed that a solution of the differential equation 3 is in the form:

$$\boldsymbol{x}=e^{\lambda t}\tilde{\boldsymbol{x}} \tag{5}$$

Here, λ denotes an eigenvalue which is a complex number in general and $\tilde{\boldsymbol{x}}$ represents the corresponding eigenvector, which may also be a complex vector. Substitution of equation 5 into differential equation 3 leads to the following standard eigenvalue problem:

$$\mathbf{A}\tilde{\boldsymbol{x}} = \lambda\tilde{\boldsymbol{x}} \tag{6}$$

which can be solved easily using commercial software such as Matlab or Mathematica. It is an easy matter to verify that the eigenvector $\tilde{\boldsymbol{x}}$ in the state space has the following structure:

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \cdots \\ \lambda\tilde{\boldsymbol{q}} \end{bmatrix} \tag{7}$$

where $\tilde{\boldsymbol{q}}$ denotes an eigenvector of the quadratic eigenvalue problem in the physical space; in other words, the $(n \times 1)$ vector $\tilde{\boldsymbol{q}}$ satisfies the equation:

$$(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})\tilde{\boldsymbol{q}} = \mathbf{0} \tag{8}$$

Eigenvalue problem formulation by Ginsberg [4]

The state vector is defined as in equation 2. The first-order matrix differential equation in the state space now reads:

$$\mathbf{S}\dot{\boldsymbol{x}}(t) - \mathbf{R}\boldsymbol{x}(t) = \mathbf{0} \tag{9}$$

where the auxiliary matrices $\mathbf{S}$ and $\mathbf{R}$ are defined as:

$$\mathbf{S} = \begin{bmatrix} -\mathbf{K} & : & \mathbf{0} \\ \cdots & : & \cdots \\ \mathbf{0} & : & \mathbf{M} \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \mathbf{0} & : & -\mathbf{K} \\ \cdots & : & \cdots \\ -\mathbf{K} & : & -\mathbf{C} \end{bmatrix} \tag{10}$$

A solution in the form of equation 5 leads to the following generalized eigenvalue problem:

$$\mathbf{R}\tilde{\boldsymbol{x}} = \lambda\mathbf{S}\tilde{\boldsymbol{x}} \tag{11}$$

where the eigenvector $\tilde{\boldsymbol{x}}$ again has the structure given in equation 7. There are $2n$ eigenvectors and corresponding $2n$ eigenvalues.

Eigenvalue problem formulation by Geradin and Rixen [5]

The state vector is defined again as in equation 2. The matrix differential equation in the state space reads:

$$\overline{\mathbf{B}}\dot{\boldsymbol{x}}(t) + \overline{\mathbf{A}}\boldsymbol{x}(t) = \mathbf{0} \tag{12}$$

where the auxiliary matrices $\overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{A}}$ are defined as:

$$\overline{\mathbf{B}} = \begin{bmatrix} \mathbf{C} & : & \mathbf{M} \\ \cdots & : & \cdots \\ \mathbf{M} & : & \mathbf{0} \end{bmatrix} \qquad \overline{\mathbf{A}} = \begin{bmatrix} \mathbf{K} & : & \mathbf{0} \\ \cdots & : & \cdots \\ \mathbf{0} & : & -\mathbf{M} \end{bmatrix} \tag{13}$$

A solution in the form of equation 5 leads to the following generalized eigenvalue problem:

$$\overline{\mathbf{A}}\tilde{\boldsymbol{x}} = -\lambda\overline{\mathbf{B}}\tilde{\boldsymbol{x}} \tag{14}$$

The eigenvector $\tilde{\boldsymbol{x}}$ in the state space has now the structure:

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \cdots \\ -\lambda\tilde{\boldsymbol{q}} \end{bmatrix} \tag{15}$$

Eigenvalue problem formulation by Frazer *et al.* [6]

The state vector in this textbook differs from the previous vectors in that it is defined as:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q} \\ \cdots \\ \mathbf{M}\dot{\boldsymbol{q}} \end{bmatrix} \tag{16}$$

In other words, 'generalized momenta' are adopted as auxiliary variables, rather than the generalized velocity vector, $\dot{\boldsymbol{q}}$.

The matrix differential equation in the state space and the corresponding system matrix are as follows:

$$\dot{\boldsymbol{x}} = \mathbf{A}^*\boldsymbol{x} \tag{17}$$

$$\mathbf{A}^* = \begin{bmatrix} \mathbf{0} & : & \mathbf{M}^{-1} \\ \cdots & : & \cdots \\ -\mathbf{K} & : & -\mathbf{C}\mathbf{M}^{-1} \end{bmatrix} \tag{18}$$

which constitute the so-called 'Hamiltonian form' [6]. A solution in the form of equation 5 leads to the standard eigenvalue problem:

$$\mathbf{A}^*\tilde{\boldsymbol{x}} = \lambda\tilde{\boldsymbol{x}} \tag{19}$$

It is easy to show that the eigenvector $\tilde{\boldsymbol{x}}$ in the state space has the following structure:

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \cdots \\ \lambda\mathbf{M}\tilde{\boldsymbol{q}} \end{bmatrix} \tag{20}$$

Eigenvalue problem formulation by Meirovitch [7]

The state vector $\boldsymbol{x}(t)$ is defined similar to equation 2 but in reversed order:

$$\boldsymbol{x} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \cdots \\ \boldsymbol{q} \end{bmatrix} \tag{21}$$

The first-order matrix differential equation in the state space reads:

$$\mathbf{M}^*\dot{\boldsymbol{x}} + \mathbf{K}^*\boldsymbol{x} = \mathbf{0} \tag{22}$$

the auxiliary matrices being

$$\mathbf{M}^* = \begin{bmatrix} \mathbf{M} & : & \mathbf{0} \\ \cdots & : & \cdots \\ \mathbf{0} & : & -\mathbf{K} \end{bmatrix} \qquad \mathbf{K}^* = \begin{bmatrix} \mathbf{C} & : & \mathbf{K} \\ \cdots & : & \cdots \\ \mathbf{K} & : & \mathbf{0} \end{bmatrix} \tag{23}$$

A solution in the form equation 5 leads to the standard eigenvalue problem:

$$\overline{\mathbf{A}}^* \tilde{\boldsymbol{x}} = \lambda \tilde{\boldsymbol{x}} \tag{24}$$

with the system matrix

$$\overline{\mathbf{A}}^* = -\mathbf{M}^{*-1}\,\mathbf{K}^* = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & : & -\mathbf{M}^{-1}\mathbf{K} \\ \cdots & : & \cdots \\ \mathbf{I} & : & \mathbf{0} \end{bmatrix} \tag{25}$$

It is easy to show that the eigenvector $\tilde{\boldsymbol{x}}$ has the following structure:

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \lambda \tilde{\boldsymbol{q}} \\ \cdots \\ \tilde{\boldsymbol{q}} \end{bmatrix} \tag{26}$$

which contains the same two ($n \times 1$) vectors as in equation 7, but in reserved order.

Eigenvalue problem formulation from Humar [8]

This textbook also makes use of the state vector given in equation 21. The matrix differential equation in the state space is:

$$\overline{\mathbf{R}}\dot{\boldsymbol{x}} + \overline{\mathbf{P}}\boldsymbol{x} = \mathbf{0} \tag{27}$$

with

$$\overline{\mathbf{R}} = \begin{bmatrix} \mathbf{0} & : & \mathbf{M} \\ \cdots & : & \cdots \\ \mathbf{M} & : & \mathbf{C} \end{bmatrix} \qquad \overline{\mathbf{P}} = \begin{bmatrix} -\mathbf{M} & : & \mathbf{0} \\ \cdots & : & \cdots \\ \mathbf{0} & : & \mathbf{K} \end{bmatrix} \tag{28}$$

A solution in the form of equation 5 leads, after some rearrangement, to a standard eigenvalue problem:

$$\mathbf{A}^{**} \tilde{\boldsymbol{x}} = \frac{1}{\lambda} \tilde{\boldsymbol{x}} \tag{29}$$

with

$$\mathbf{A}^{**} = -\overline{\mathbf{P}}^{-1}\overline{\mathbf{R}} = \begin{bmatrix} \mathbf{0} & : & \mathbf{I} \\ \cdots & : & \cdots \\ -\mathbf{K}^{-1}\mathbf{M} & : & -\mathbf{K}^{-1}\mathbf{C} \end{bmatrix} \tag{30}$$

It is worth nothing that the results of the solution of the eigenvalue problem yield the inverses of the eigenvalues of the mechanical system.

Numerical applications

Consider the two-degree-of-freedom vibrational system shown in Fig. 1, which is taken from Shabana [9]. The mass, damping and stiffness matrices of the system are, respectively:

$$\mathbf{M}=\begin{bmatrix} m & 0 \\ 0 & J_{\mathrm{G}} \end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix} c_1+c_2 & c_2 b-c_1 a \\ c_2 b-c_1 a & c_1 a^2+c_2 b^2 \end{bmatrix}$$

$$\mathbf{K}=\begin{bmatrix} k_1+k_2 & k_2 b-k_1 a \\ k_2 b-k_1 a & k_1 a^2+k_2 b^2 \end{bmatrix}$$

where m and J_G denote the mass and moment of inertia of the bar with respect to the axis through the center of mass, G, respectively.

The choice of the physical parameters as m = 1000 kg, L = 4 m, $a = b = L/2 = 2$ m, J_G = 1300 kg m^2, $k_1 = 50.10^3$ N/m, $k_2 = 70.10^3$ N/m; $c_1 = c_2 = 10$ Ns/m leads to the following matrices:

$$\mathbf{M}=\begin{bmatrix} 1000 & 0 \\ 0 & 1300 \end{bmatrix} \qquad \mathbf{C}=\begin{bmatrix} 20 & 0 \\ 0 & 80 \end{bmatrix} \qquad \mathbf{K}=\begin{bmatrix} 120{,}000 & 40{,}000 \\ 40{,}000 & 480{,}000 \end{bmatrix}$$

Solution of the eigenvalue problem defined by equations 6 and 4 via Matlab gives the following results for the four eigenvalues and eigenvectors:

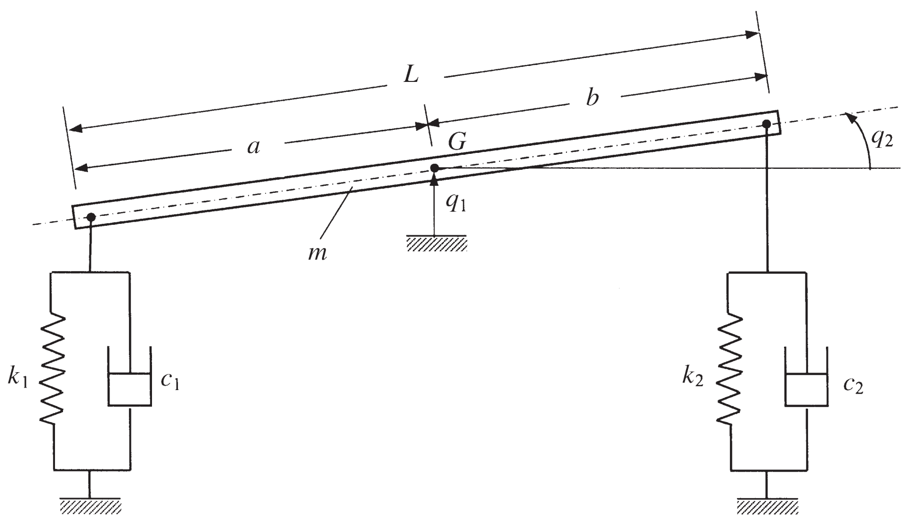

Fig. 1 *Sample vibrational system with two degrees of freedom.*

$$\lambda_{1,2} = -0.01038857 \pm 10.73106641i$$
$$\lambda_{3,4} = -0.03038066 \pm 19.34099068i \tag{a}$$

$$\tilde{\boldsymbol{x}}_{1,2} = \begin{bmatrix} 1.00000000 \\ -0.12110285 \pm 0.00020849i \\ \cdots\cdots\cdots\cdots \\ -0.01038857 \pm 10.73106641i \\ -0.00097923 \mp 1.29956486i \end{bmatrix}$$
$$\tilde{\boldsymbol{x}}_{3,4} = \begin{bmatrix} 1.00000000 \\ 6.35184013 \pm 0.01970911i \\ \cdots\cdots\cdots\cdots \\ -0.03038066 \pm 19.34099068i \\ -0.57416673 \pm 122.85028188i \end{bmatrix} \tag{b}$$

In accordance with the structure of the eigenvector $\tilde{\boldsymbol{x}}$ given in equation 7, the upper parts of (b) represent the eigenvectors $\tilde{\boldsymbol{q}}$ in the physical space, whereas the lower parts represent the vector $\tilde{\boldsymbol{q}}$ multiplied by the eigenvalue, λ. That the (3,1) elements of the eigenvectors in (b) are the same as the eigenvalues in (a) reflects the above fact.

The generalized eigenvalue problem defined by equation 11 (and 10) gives exactly the same results as in (a) and (b).

The generalized eigenvalue problem defined by equation 14 (and 13) yields the same results, recognizing that the eigenvalues in equation 14 are $-\lambda$, rather than λ.

The standard eigenvalue problem defined by equation 19 (and 18) gives the same eigenvalues as in (a). The eigenvectors are now as follows:

$$\tilde{\boldsymbol{x}}_{1,2} = \begin{bmatrix} 1.00000000 \\ -0.12110285 \pm 0.00020849i \\ \cdots\cdots\cdots\cdots \\ -10.38857206 \pm 10731.06640786i \\ -1.27299949 \mp 1689.43432437i \end{bmatrix}$$
$$\tilde{\boldsymbol{x}}_{3,4} = \begin{bmatrix} 1.00000000 \\ 6.35184013 \pm 0.01970911i \\ \cdots\cdots\cdots\cdots \\ -30.38065871 \pm 19340.99067774i \\ -746.41675199 \pm 159705.36644497i \end{bmatrix} \tag{c}$$

The upper parts of (c) give the eigenvectors $\tilde{\boldsymbol{q}}$ in the physical space, whereas those in the lower parts represent these eigenvectors multiplied by $\lambda\mathbf{M}$, in accordance with the structure in equation 20.

The solution of the standard eigenvalue problem in equation 24 with the matrix in equation 25 yields the same eigenvalues given in (a). The eigenvectors are as follows:

$$\tilde{\boldsymbol{x}}_{1,2} = \begin{bmatrix} -0.01038857 \pm 10.73106641i \\ -0.00097923 \mp 1.29956486i \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 1.00000000 \\ -0.12110285 \pm 0.00020849i \end{bmatrix}$$

$$\tilde{\boldsymbol{x}}_{3,4} = \begin{bmatrix} -0.03038066 \pm 19.34099068i \\ -0.57416673 \pm 122.85028188i \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 1.00000000 \\ 6.35184013 \pm 0.01970911i \end{bmatrix} \tag{d}$$

The lower parts represent the eigenvectors $\tilde{\boldsymbol{q}}$ in the physical space, whereas the upper parts correspond to $\lambda\tilde{\boldsymbol{q}}$, in accordance with the structure in equation 26.

The eigenvalues of the standard eigenvalue problem defined by equation 29 are obtained as:

$$\frac{1}{\lambda_{1,2}} = -0.01038857 \pm 10.73106641i$$

$$\frac{1}{\lambda_{3,4}} = -0.03038066 \pm 19.34099068i \tag{e}$$

whereas the eigenvectors are the same as in (d).

Conclusions

Various eigenvalue problem formulations for the vibrations of viscously damped linear mechanical systems have been examined, relationships between them have been established and results have been applied to a numerical example on a two-degree-of-freedom vibrational system.

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