An alternative shear formula in non-principal coordinates

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Abstract An alternative formula is presented to compute shear stresses in non-principal coordinates of arbitrary cross-sections for linear elastic, prismatic beams. In contrast to the formula traditionally used in textbooks for non-principal coordinates, the new formula avoids lengthy computations with the first and second moments of area. The trade-off is that the normal stresses *must* be computed before the shear stresses; however, since in practical situations both normal stresses and shear stresses need to be known, this is not considered a general disadvantage of the proposed formula. Whereas the traditional formula takes different formats for principal and non-principal coordinates, the new formula is valid in any coordinate system. Thus, simplicity is gained.

Keywords shear formula; prismatic beams; beam theory; thin-walled structures

Introduction

In many engineering applications, shear stresses have to be computed for arbitrary cross-sections. The traditional formulas for the computation of shear stresses appear in two formats. The first, simplest format requires that the principal axes of the cross-section be known and used. This is considered as a limitation, since the transformation from an arbitrary coordinate system to the principal coordinates can be lengthy and burdensome. The second, more lengthy format is an extension of the formula for principal coordinates, and requires the decomposition of the first and second moments of area.

The limitations of the traditional formulas for shear stresses have been recognized by Wang [1], who proposed vector formulas as an alternative to overcome the restriction to the principal axes. Although the derivations in [1] do not depend on the coordinate system, a basic knowledge of vector algebra is needed. Here, we present an alternative formula for the computation of shear stresses, which does not relate to the principal axes, and nor does it require any knowledge of vector algebra. As such, the proposed formula is well suited for education of engineering students, as well as for use in engineering practice.

The proposed formula uses the same starting point as the traditional formula for shear stresses: the equilibrium of a *fraction* of the cross-section, thus relating the shear flow to the part of the normal stresses that are due to bending. In order to introduce all quantities and symbols in a systematic manner, this well known part of the theory is briefly reviewed in the next section. The traditional shear stress formula assumes a coordinate system in the principal directions to arrive at transparent, easy-to-use formulas – but for the case of non-principal coordinates the complexity of the formula increases. Here, an alternative formula is derived that does not require coordinate axes in the principal directions. Nonetheless, the resulting formula to compute the shear stress is of a similar simplicity to the traditional formula for principal coordinates. The next two sections deal with the traditional formulas and the newly proposed formula, respectively, where both depart from the well known formula for shear flow. The subsequent section illustrates how the new shear formula can be used.

Shear flow in prismatic beams

In Fig. 1, part of a prismatic beam is considered, of length Δx. The origin of the orthonormal coordinate system corresponds to the centroid of the cross-section, where the beam is aligned along the x axis, while the y axis and the z axis have arbitrary orientations. Distributed loads q_y and q_z act on the beam in the y and z directions. For simplicity in the following derivations, no distributed load in the x direction is considered: $q_x = 0\,\text{kN/m}$. Furthermore, it is assumed that Young's modulus, E, is constant within the cross-section. In Fig. 1, the positive shear forces, V, and bending moments, M, are depicted. Since $q_x = 0$, no change of normal forces occurs, and the normal forces have been left out of consideration in the following.

With the introduced sign conventions for V and M, the following well known differential relations can be derived:

$$V_y = \frac{\mathrm{d}M_z}{\mathrm{d}x} \tag{1}$$

$$V_z = \frac{\mathrm{d}M_y}{\mathrm{d}x} \tag{2}$$

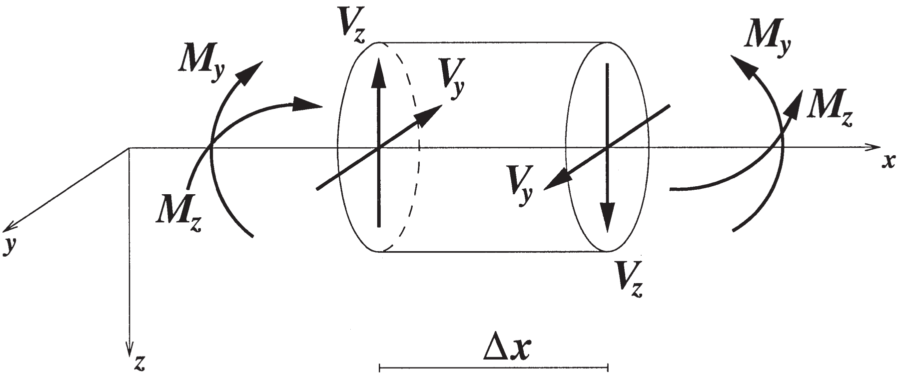

Fig. 1. *Notation and sign conventions.*

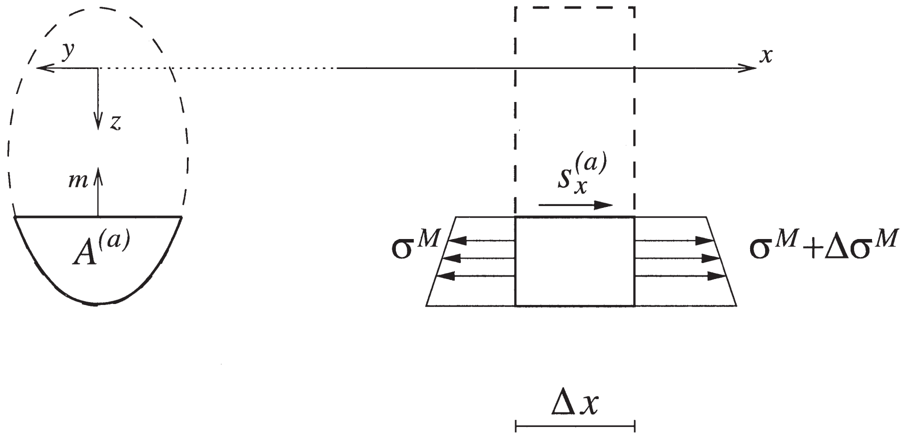

Fig. 2. *Definition of* $A^{(a)}$ *and normal vector,* **m**, *in a cross-sectional cut (left) and shear flow along a longitudinal cut (right).*

The presence of the bending moments, M_y and M_z, results in a distribution of normal stresses. When only a portion of the cross-section is considered (see Fig. 2), force equilibrium in the x direction reveals that:

$$-R_M^{(a)} + s_x^{(a)}\Delta x + R_M^{(a)} + \Delta R_M^{(a)} = 0 \tag{3}$$

where R_M is the resultant of the bending-induced normal stresses $\sigma^M = \sigma^M_{xx}$, and s_x is the shear flow (force per unit of length) in the x direction. The superscript (a) denotes the portion of the cross-section that is considered separately; $R_M^{(a)}$ is the resultant of the normal stresses that act on the portion (a) of the cross-section, or:

$$R_M^{(a)} = \int_{A^{(a)}} \sigma^M(y, z)\mathrm{d}A \tag{4}$$

Furthermore, a vector, $\boldsymbol{m}$, can be defined that is normal to the cut, which is needed for a unique definition of the shear stresses (see below).

Equation 3 leads to the well established differential relation between shear flow and the resultant of the normal stresses:

$$s_x^{(a)} = -\frac{\mathrm{d}R_M^{(a)}}{\mathrm{d}x} \tag{5}$$

Equation 5 does not present an expression which can easily be used in practice, and therefore further derivations are necessary. Equation 5 is used as the starting point for both the traditional and the alternative formulas.

Traditional formulas

Traditionally, the derivation continues with the assumption that y and z are the principal directions of the cross-section. If this is the case, the normal stress due to bending, $\sigma^M(y, z)$, can be written as:

$$\sigma^M(y,z) = \frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{6}$$

where I_y and I_z are the second moments of area of the cross-section. Equations 4–6 and expressions 1 and 2 lead to:

$$s_x^{(a)} = -\int_{A^{(a)}} \frac{\mathrm{d}\sigma^M(y,z)}{\mathrm{d}x}\mathrm{d}A = -\frac{V_y}{I_z}\int_{A^{(a)}} y\mathrm{d}A - \frac{V_z}{I_y}\int_{A^{(a)}} z\mathrm{d}A \tag{7}$$

or

$$s_x^{(a)} = -\frac{V_y Q_z^{(a)}}{I_z} - \frac{V_z Q_y^{(a)}}{I_y} \tag{8}$$

where $Q_y^{(a)}$ and $Q_z^{(a)}$ are the first moments of area of the separated portion of the cross-section. The shear stress, τ_{mx}, is assumed to be constant over the width of the cross-section, $b^{(a)}$:

$$\tau_{mx} = \frac{s_x^{(a)}}{b^{(a)}} \tag{9}$$

Finally, equivalence of shear stresses in two orthogonal planes yields:

$$\tau_{xm} = \tau_{mx} = \frac{s_x^{(a)}}{b^{(a)}} = -\frac{V_y Q_z^{(a)}}{I_z b^{(a)}} - \frac{V_z Q_y^{(a)}}{I_y b^{(a)}} \tag{10}$$

Note that direction of the shear stress, τ_{xm}, is always related to the separated portion of the cross-section that is considered: a positive shear stress is directed in the positive m direction, which orientation is set by the cut that is made.

If the restriction to principal coordinates is dropped, equation 6 is expanded as:

$$\sigma^M(y,z) = \frac{(M_z I_y - M_y I_{yz})y}{I_y I_z - I_{yz}^2} + \frac{(M_y I_z - M_z I_{yz})z}{I_y I_z - I_{yz}^2} \tag{11}$$

Then, an expression for the shear stresses can be found as (see for instance [2]):

$$\tau_{xm} = -\frac{(V_y I_y - V_y I_{yz})Q_z^{(a)}}{(I_y I_z - I_{yz}^2)b^{(a)}} - \frac{(V_z I_z - V_y I_{yz})Q_y^{(a)}}{(I_y I_z - I_{yz}^2)b^{(a)}} \tag{12}$$

which is considerably more complex than equation 10.

Alternative formula

As an alternative, for the derivative $\mathrm{d}R_M^{(a)}/\mathrm{d}x$ from equation 5, the chain rule can be applied as:

$$\frac{\mathrm{d}R_M^{(a)}}{\mathrm{d}x} = \frac{\mathrm{d}R_M^{(a)}}{\mathrm{d}M}\frac{\mathrm{d}M}{\mathrm{d}x} = \frac{\mathrm{d}R_M^{(a)}}{\mathrm{d}M}V \tag{13}$$

in which no assumption regarding the coordinate axes system has been made. Due to the linear dependence of $R_M^{(a)}$ on M, it can be written that [3]:

$$\frac{\mathrm{d}R_M^{(a)}}{\mathrm{d}M} = \frac{R_M^{(a)}}{M} \tag{14}$$

Thus, the shear flow can be elaborated as:

$$s_x^{(a)} = -\frac{R_M^{(a)}}{M}V \tag{15}$$

and the shear stress is again assumed to be constant over the width of the separated portion:

$$\tau_{xm} = -\frac{R_M^{(a)}}{M}\frac{V}{b^{(a)}} \tag{16}$$

In our proposed methodology, equations 15 and 16 replace equations 8 and 10, respectively. When these two sets of equations are compared, it can be seen that the ratio $R_M^{(a)}/M$ plays the role of $Q_y^{(a)}/I_y$ or of $Q_z^{(a)}/I_z$. A few remarks can be made.

First, since $Q_y^{(a)}/I_y$ and $R_M^{(a)}/M$ are equivalent, $R_M^{(a)}/M$ can be interpreted as a *geometrical* (rather than *mechanical*) property of the cross-section. As such, it is independent of V in the sense that there does not need to be a mechanical relation between V and M. Although it may seem counter-intuitive at first glance, *any fictitious bending moment M may be considered*, as long as M acts in the same plane as V and as long as $R_M^{(a)}$ is computed in accordance with the chosen M. The reason why M and V should act in the same plane can be seen from equation 13: application of the chain rule and substitution of V is valid only if V is the derivative of M, that is, if M and V act in the same plane.

Second, in the derivation of equations 15 and 16, no assumptions regarding the coordinate system have been made. Indeed, equations 15 and 16 are valid in any coordinate system. Thus, the same simple formulas can be used for principal and non-principal coordinates, which is in contrast to the traditional approach.

Third, in order to compute the shear flow or the shear stress at a particular point, a bending moment must be assumed, and the resultant force, $R_M^{(a)}$, needs to be known. Therefore, an arbitrary normal stress distribution must be determined before evaluation of the shear stresses. This could be considered a disadvantage of the proposed method. However, normally both the normal stress distribution and the shear stress distribution are needed, so that the actual (rather than a fictitious) normal stress distribution can be used to compute $R_M^{(a)}/M$.

Fourth, equations 15 and 16 show that shear stresses are computed by considering the resultant of bending-induced normal stresses on a portion of the cross-section. The same consideration has been used in the *derivation* of the shear formulas. Thus, a close relation between the *derivation* and the *application* of the shear formulas is established, which improves insight into the shear formulas.

Finally, as can be easily seen from equations 4 and 16, the bending-induced normal stresses are proportional to the x derivative of the shear stresses. Thus, if σ^M is constant in a part of the cross-section with constant thickness, then τ is linear in that part, and if σ^M is linear, then τ is quadratic. Furthermore, τ takes its maximum or minimum value where the neutral axis cuts the cross-section, i.e. where $\sigma^M = 0$. These particularities can be used to sketch a shear stress diagram quickly, after which the shear stress has to be computed only at a few relevant points.

Summarizing, equations 15 and 16 can be used in the following manner:

(1) A coordinate system should be selected, one axis of which should correspond to the direction of the applied shear force, V. This does not need to be the principal coordinate system.
(2) A bending moment, M, should be selected in the plane of V. When distributions of both normal stresses and shear stresses are required, it may be most efficient not to use a fictitious M, but the actual M.
(3) For the selected M, the ratio $R_M^{(a)}/M$ can be determined as a continuous function or for a few relevant points.
(4) With the classical sign conventions that relate shear flow and shear stress to the shear force, the shear stress distribution can be obtained.

The next section shows how the proposed formula can be used in engineering practice.

Examples

Shear stress computed as a continuous function

As a first example, a thin-walled L-shaped cross-section is studied, of which the geometry is given in Fig. 3. The thickness of both web and flange is t. The second moments of area are given as $I_y = 36ta^3$, $I_z = \frac{27}{4}ta^3$ and $I_{yz} = -9ta^3$. A vertical shear force, V_z, is present, while $V_y = 0$. Corresponding to V_z, a bending moment, M_y, is considered. The normal stress distribution that is in accordance with M_y can be elaborated as:

$$\sigma^M(y,z) = \frac{M_y}{ta^3}\left(\frac{1}{18}y + \frac{1}{24}z\right) \tag{17}$$

so that for the neutral axis $z = -4y/3$ is found (see Fig. 3).

First, the flange is considered. For the ratio $R_M^{(a)}/M$ it can be written that:

$$\frac{R_M^{(a)}}{M} = \frac{1}{M_y}\int_{A^{(a)}} \sigma(y,z)\mathrm{d}A = \frac{1}{ta^3}\int_{A^{(a)}}\left(\frac{1}{18}y + \frac{1}{24}z\right)\mathrm{d}A \tag{18}$$

Since for the flange it holds that $z = -2a$, equation 18 can be elaborated as:

$$\frac{R_M^{(a)}}{M} = \frac{1}{ta^3}\int_{\tilde{y}}^{5a/2}\left(\frac{1}{18}y - \frac{a}{12}\right)t\mathrm{d}y = \frac{-5a^2 + 12a\tilde{y} - 4\tilde{y}^2}{144a^3} \tag{19}$$

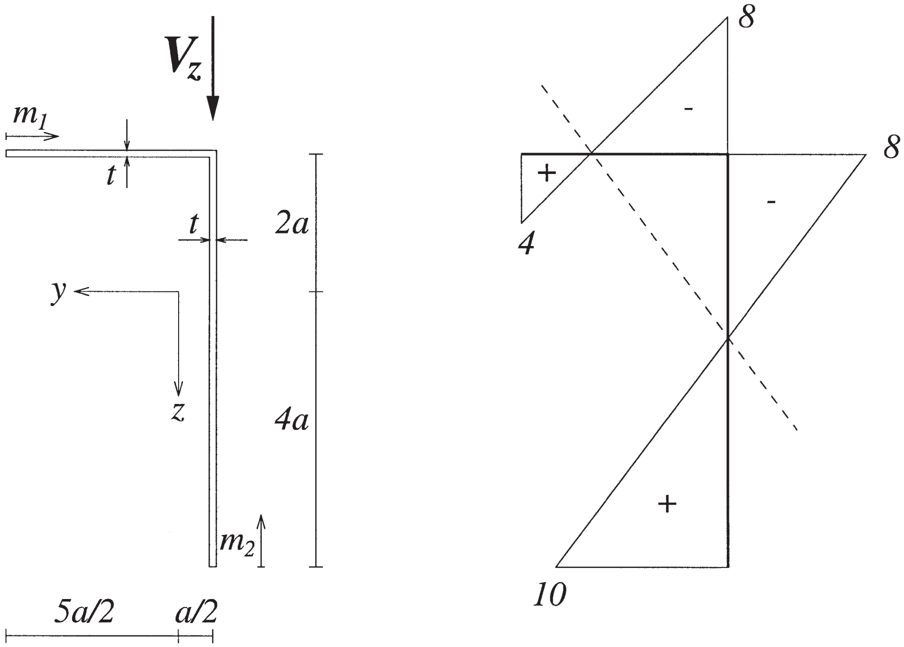

Fig. 3. *L-shape example: geometry (left) and normal stress distribution expressed in terms of* $M_y/72ta^2$ *(right); neutral axis is indicated with dashed line.*

where $\tilde{y}$ is the coordinate of the cut. Changing coordinates via $\tilde{y} = -m_1 + 5a/2$ (where m_1 is the distance measured from the left tip of the flange – see Fig. 3), an expression for the shear stress is found as:

$$\text{flange:} \quad \tau_{xm} = -\frac{R_M^{(a)}}{M}\frac{V_z}{t} = \frac{V_z}{144ta^3}\left(4m_1^2 - 8am_1\right) \tag{20}$$

Following similar lines, for the web one can write:

$$\frac{R_M^{(a)}}{M} = \frac{1}{M_y}\int_{\tilde{z}}^{4a} \frac{M_y}{ta^3}\left(-\frac{a}{36} + \frac{z}{24}\right) t\,\mathrm{d}z = \frac{32a^2 + 4a\tilde{z} - 3\tilde{z}^2}{144a^3} \tag{21}$$

where $\tilde{z}$ is the coordinate of the cut. Changing coordinates via $\tilde{z} = -m_2 + 4a$ (where m_2 is the distance measured from the bottom tip of the web – see Fig. 3) leads to an expression for the shear stress as:

$$\text{web:} \quad \tau_{xm} = -\frac{R_M^{(a)}}{M}\frac{V_z}{t} = \frac{V_z}{144ta^3}\left(3m_2^2 - 20am_2\right) \tag{22}$$

In Fig. 4, the shear stress distribution is plotted for the complete cross-section, expressed in terms of $V_z/144ta$. As can be verified, the shear stress assumes its maximum values where the neutral axis cuts the cross-section.

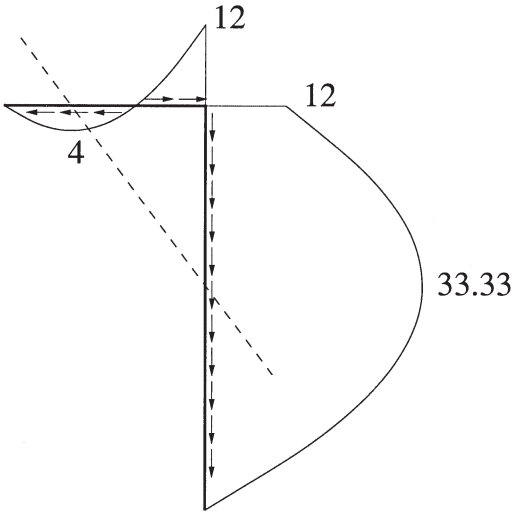

Fig. 4. *L-shape example: shear stress distribution in terms of* $V_z/144ta$*; neutral axis is indicated with dashed line.*

Note that equations 19 and 21 contain only geometric quantities. As mentioned above, the ratio $R_M^{(a)}/M$ is a purely geometric quantity. This explains why any fictitious bending moment, M, can be used for the evaluation of $R_M^{(a)}/M$.

Shear stress computed in discrete points

As mentioned above, it is equally possible to compute the shear stress not as a continuous function, but in a limited set of relevant points, using *a priori* knowledge of the shear stress diagram (linear where σ^M is constant, quadratic where σ^M is linear, extremum where $\sigma^M = 0$). This approach will be demonstrated by means of the asymmetric thin-walled T-shaped cross-section shown in Fig. 5. With the given dimensions and the origin of the coordinate axes in the centroid of the cross-section, $I_y = 34.56 \times 10^6\,\text{mm}^4$, $I_z = 16.128 \times 10^6\,\text{mm}^4$ and $I_{yz} = -6.912 \times 10^6\,\text{mm}^4$.

A fictitious bending moment is taken, M_y, corresponding to the applied shear force, which leads to a normal stress distribution as:

$$\sigma^M(y, z) = M_y\left\{\left(135.6 \times 10^{-6}\,\text{mm}^{-4}\right)y + \left(316.5 \times 10^{-6}\,\text{mm}^{-4}\right)z\right\} \tag{23}$$

so that for the neutral axis it is found that $z = -3y/7$. The normal stress distribution and neutral axis are plotted in Fig. 5. The relevant points of the shear stress diagram are designated A–E in Fig. 5.

At points A, C and D, the shear stress equals zero. Since the neutral axis cuts the cross-section at point A, the slope of the shear stress diagram is zero at A (note that the normal stress is proportional to the derivative of the shear stress). The shear stress has a non-zero slope at C and D.

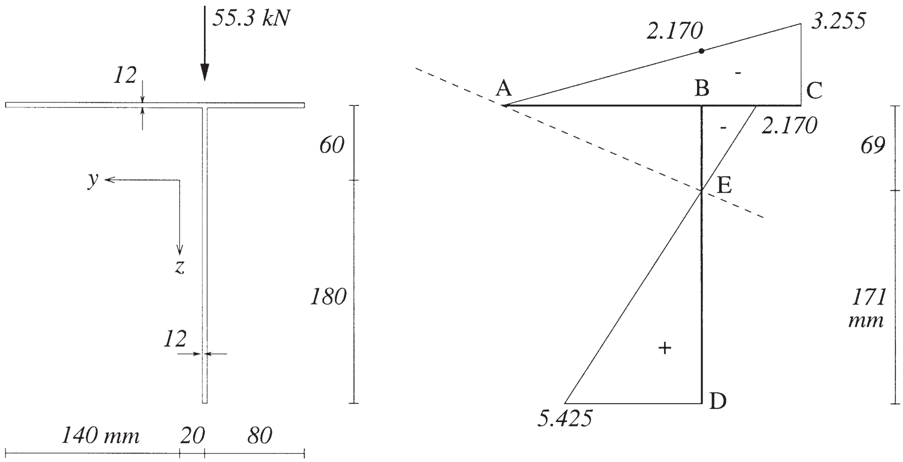

Fig. 5. *T-shape example: geometry (left) and normal stress distribution expressed in terms of* $M_y \times 10^{-6}\,mm^{-3}$ *(right); neutral axis is indicated with dashed line.*

The shear stress at E can be found by integrating the normal stress from D to E (which implies a positive *m* direction from D to E), such that:

$$\frac{R_M^{(a)}}{M} = \frac{\frac{1}{2}\left(5.425 \times M_y \times 10^{-6}\,\mathrm{mm}^{-3}\right) \times 171\,\mathrm{mm} \times 12\,\mathrm{mm}}{M_y} = 5.566 \times 10^{-3}\,\mathrm{mm}^{-1} \quad (24)$$

Then, the shear stress at point E can be elaborated as:

$$\tau_{xm} = -\frac{R_M^{(a)}}{M}\frac{V}{b^{(a)}} = -5.566 \times 10^{-3}\,\mathrm{mm}^{-1} \times \frac{55300\,\mathrm{N}}{12\,\mathrm{mm}} = -25.6\,\mathrm{N/mm}^2 \quad (25)$$

where the negative sign indicates that the shear stress acts in the negative *m* direction, i.e. in the positive *z* direction. In a similar manner, the shear stress in the web at point B is found via:

$$\frac{R_M^{(a)}}{M} = \frac{\frac{1}{2}\left\{(5.425 - 2.170) \times M_y \times 10^{-6}\,\mathrm{mm}^{-3}\right\} \times 240\,\mathrm{mm} \times 12\,\mathrm{mm}}{M_y}$$

$$= 4.687 \times 10^{-3}\,\mathrm{mm}^{-1} \quad (26)$$

$$\tau_{xm} = -4.687 \times 10^{-3}\,\mathrm{mm}^{-1} \times \frac{55300\,\mathrm{N}}{12\,\mathrm{mm}} = -21.6\,\mathrm{N/mm}^2 \quad (27)$$

For the left part of the flange, the normal stress diagram between A and B is considered, so that:

$$\frac{R_M^{(a)}}{M} = \frac{\frac{1}{2}\left(-2.170 \times M_y \times 10^{-6}\,\mathrm{mm}^{-3}\right) \times 160\,\mathrm{mm} \times 12\,\mathrm{mm}}{M_y}$$

$$= -2.083 \times 10^{-3}\,\mathrm{mm}^{-1} \quad (28)$$

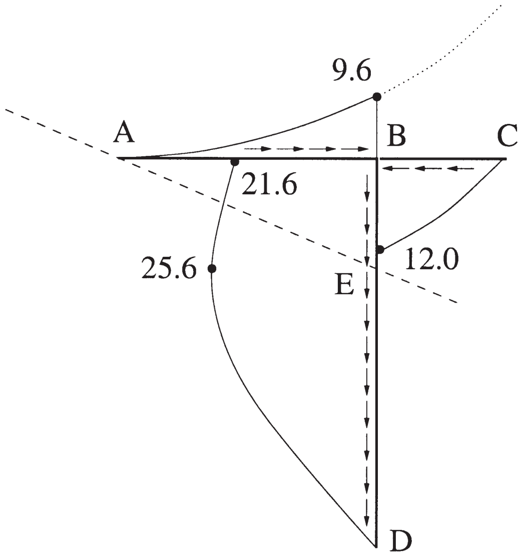

Fig. 6. *T-shape example: shear stress distribution in N/mm^2; neutral axis is indicated with dashed line; dotted line denotes the continuation of shear stress from* AB *to* BC.

$$\tau_{xm} = -\left(-2.083\times10^{-3}\,\text{mm}^{-1}\right)\times\frac{55300\,\text{N}}{12\,\text{mm}} = 9.6\,\text{N/mm}^2 \tag{29}$$

where the positive sign indicates that the shear stress acts in the positive m direction (from A to B), i.e. in the negative y direction for this part of the flange. For the right part of the flange, the normal stress distribution between C and B leads to:

$$\frac{R_M^{(a)}}{M} = \frac{\frac{1}{2}\left\{(-3.255-2.170)\times M_y\times10^{-6}\,\text{mm}^{-3}\right\}\times80\,\text{mm}\times12\,\text{mm}}{M_y}$$
$$= -2.604\times10^{-3}\,\text{mm}^{-1} \tag{30}$$

$$\tau_{xm} = -\left(-2.604\times10^{-3}\,\text{mm}^{-1}\right)\times\frac{55300\,\text{N}}{12\,\text{mm}} = 12.0\,\text{N/mm}^2 \tag{31}$$

where the positive sign indicates that the shear stress acts in the positive m direction (from C to B), which is now in the positive y direction. In Fig. 6 the complete shear stress diagram is plotted. It can be verified that the total shear flow towards B equals the total shear flow from B. The dotted line in Fig. 6 denotes a fictitious shear stress distribution *without* discontinuity in B, i.e. a continuation of the parabolic distribution in AB. The actual shear stress distribution in BC is found by translating this fictitious distribution over 21.6 N/mm^2.

Note that the determination of shear stresses in arbitrary thin-walled tubes requires that an axial cut is made in the cross-section, after which the shear stress is deter-

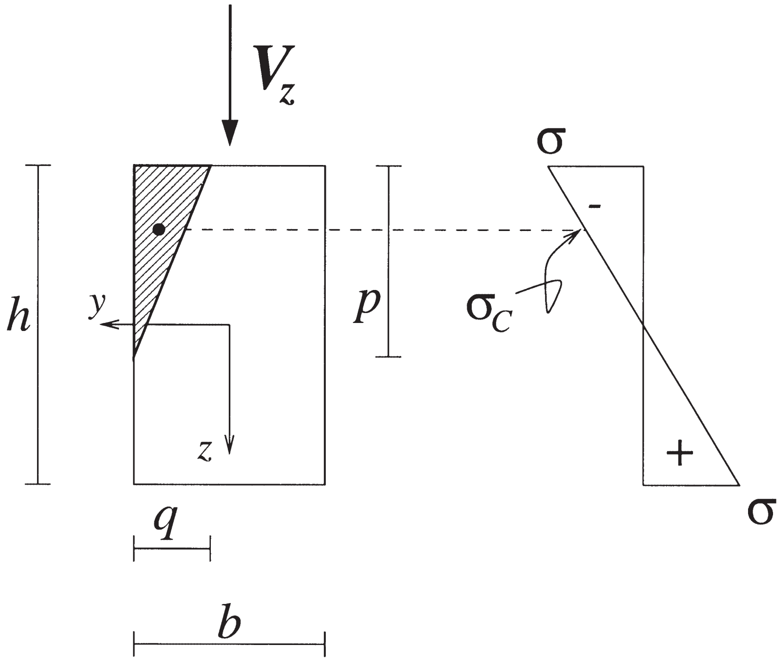

Fig. 7. *Thick-walled example: geometry (left) and normal stress distribution (right); triangular portion with dimensions* p *and* q.

mined for the open cross-section and shear deformation due to torques is applied to calculate the shear stresses in the original tubular cross-section. The problem statement for the opened cross-section is similar to that of a non-tubular cross-section; therefore, shear stress distributions in tubular cross-sections can also be computed efficiently with the newly proposed formula.

Shear flow in a thick-walled section

Although possible applications will probably concentrate on thin-walled structures, it is emphasized that equation 15 is valid for thin-walled structures as well as for thick-walled structures, while equation 16 holds for thick-walled structures in an approximate sense. Below, the shear flow along skew sections in a rectangular section is analysed. Based on the shear flow, the shear stress can be estimated by assuming a constant distribution.

In Fig. 7, a rectangular cross-section is plotted with a vertical shear force, V_z. The normal stress distribution due to a fictitious bending moment, M, is also given, whereby the extreme values are denoted as σ. A triangular portion of the cross-section is considered, with base q and height p (see Fig. 7). The value of the normal stress at the centroid of the triangle is denoted as σ_C; it can be verified that:

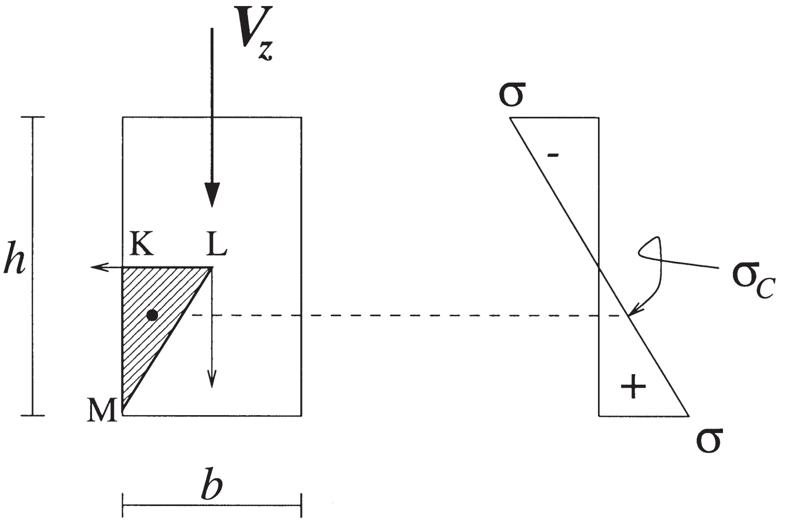

Fig. 8. *Thick-walled example: geometry (left) and normal stress distribution (right); triangular portion with dimensions* $\frac{1}{2}$h *and* $\frac{1}{2}$b.

$$\sigma_C = -\sigma \frac{\frac{1}{2}h - \frac{1}{3}p}{\frac{1}{2}h} = -\sigma \frac{3h-2p}{3h} \tag{32}$$

Since $R_M^{(a)} = \frac{1}{2}pq \times \sigma_C$ and $M = \frac{1}{6}bh^2\sigma$, for the shear flow it follows that:

$$s_x^{(a)} = -\frac{R_M^{(a)}}{M}V_z = -\frac{\frac{1}{2}pq\sigma_C}{\frac{1}{6}bh^2\sigma}V_z = \frac{1}{h}\left(3-2\frac{p}{h}\right)\frac{p}{h}\frac{q}{b}V_z \tag{33}$$

In particular, if $p = h$ and $q = b$, then $s_x^{(a)} = V_z/h$.

This result can be used to verify the shear flow on the triangular cross-section shown in Fig. 8. Here, it holds that $\sigma_C = \frac{1}{3}\sigma$; therefore $R_M^{(a)} = \frac{1}{24}bh\sigma$, and

$$\text{KLM:}\quad s_x^{(a)} = -\frac{R_M^{(a)}}{M}V_z = -\frac{\frac{1}{24}bh\sigma}{\frac{1}{6}bh^2\sigma}V_z = -\frac{V_z}{4h} \tag{34}$$

This is the *total* shear flow acting on the shaded portion of the cross-section, corresponding to the horizontal cut KL as well as to the diagonal cut LM. The negative sign indicates that this shear flow is directed in the negative x direction (cf. Fig. 2). It is well known that the maximum shear *stress*, τ, in a rectangular thick-walled section equals $\frac{3}{2}\frac{V}{bh}$ and that it occurs along the y axis of this cross-section. Therefore, the shear flow along KL (cf. Fig. 9) is:

$$\text{KL:}\quad s_x^{(a)} = -\frac{3}{2}\frac{V_z}{bh} \times \frac{1}{2}b = -\frac{3V_z}{4h} \tag{35}$$

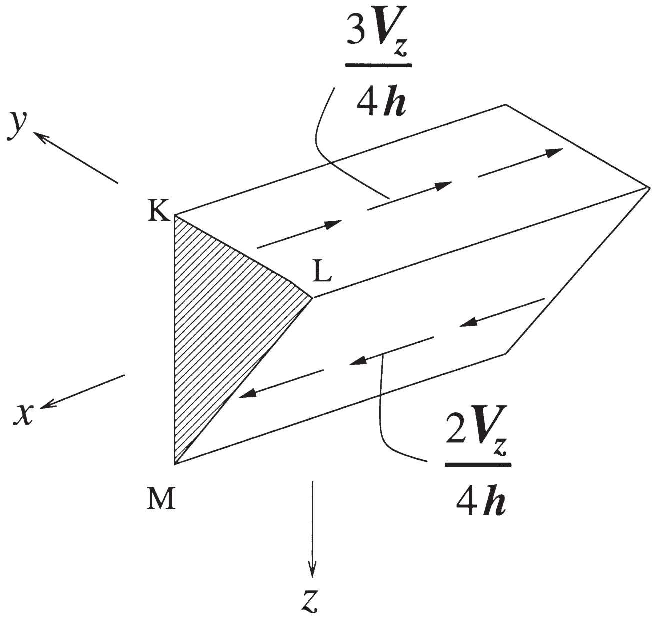

Fig. 9. *Thick-walled example: shear flow along cuts KL and LM.*

Comparing equations 34 and 35, for cut LM it is found that:

$$\text{LM:}\quad s_x^{(a)} = +\frac{2V_z}{4h} \tag{36}$$

where the positive sign indicates that the shear flow along LM is directed in the positive x direction. Indeed, the shear flow found for LM is half the shear flow found with equation 33, taking $p = h$ and $q = b$.

To show that the same theoretical framework can also be used without reference to the principal coordinates, the same rectangular cross-section is considered with a shear force that is oriented along a diagonal of the cross-section (Fig. 10). The normal stress is given by equation 6, by which it can be verified that the neutral axis is directed along the other diagonal of the cross-section. To link the (fictitious) bending moment M to the normal stress, it must be realized that the three-dimensional normal stress diagram of the cross-section consists of two tetrahedrons, one corresponding to the tensile stresses and one corresponding to the compressive stresses. For the volume of a tetrahedron it holds that:

$$\text{volume tetrahedron} = \frac{1}{3} \times \text{base} \times \text{height} = \frac{1}{3} \times \frac{1}{2} bh \times \sigma = \frac{1}{6} bh\sigma \tag{37}$$

Furthermore, the centroid of a tetrahedron is located a quarter of the distance from a vertex to the centroid of the opposite face. Thus, the two centroids (one for the

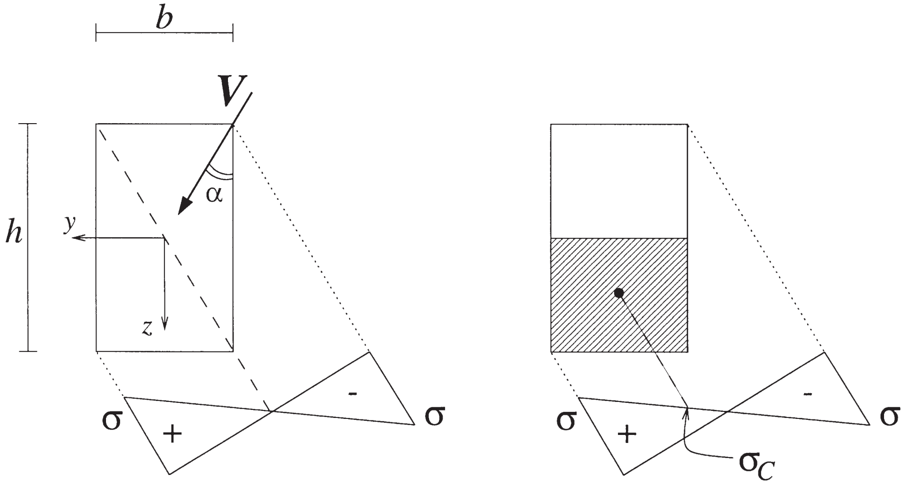

Fig. 10. *Thick-walled example with inclined shear force; neutral axis indicated with dashed line.*

tensile stress, one for the compressive stress) are located at $(y, z) = (\tfrac{1}{4}b, \tfrac{1}{4}h)$ and $(y, z) = (-\tfrac{1}{4}b, -\tfrac{1}{4}h)$, respectively. The distance between the centroids of the two tetrahedrons is then given by $\tfrac{1}{2}\sqrt{h^2+b^2}$. Now the bending moment, M, can be expressed as:

$$M = \text{volume tetrahedron} \times \text{distance between centroids} = \frac{1}{12} bh\sigma\sqrt{h^2+b^2} \tag{38}$$

Next, the lower half of the cross-section (see Fig. 10, right) will be considered to compute the shear flow. The bending stress resultant on this portion is given by:

$$R_M^{(a)} = \frac{1}{2} bh \times \sigma_C \tag{39}$$

where $\sigma_C = \tfrac{1}{4}\sigma$. Thus,

$$s_x^{(a)} = -\frac{R_M^{(a)}}{M} V = -\frac{\tfrac{1}{8} bh\sigma}{\tfrac{1}{12} bh\sigma\sqrt{h^2+b^2}} V = -\frac{3}{2\sqrt{h^2+b^2}} V \tag{40}$$

Since the vertical component of V can be expressed as $V_z = Vh/\sqrt{h^2+b^2}$, the well known expression

$$s_x^{(a)} = -\frac{3}{2}\frac{V_z}{h} \tag{41}$$

is obtained.

Conclusion

In this contribution, an alternative formula has been proposed, derived and illustrated to compute shear flows and shear stresses in prismatic beams of arbitrary cross-section. The transformation to principal coordinates is completely avoided. In contrast to the traditional formulas, which take a simple format for principal coordinates and a complicated format for non-principal coordinates, in the proposed approach the same simple formula can be used for principal and non-principal coordinates. Furthermore, in the proposed formulas the computation of shear stresses is closely linked to the underlying derivation of the shear formulas, and therefore insight is increased. As a trade-off, a normal stress distribution must be computed before the shear stresses can be determined. However, since in engineering practice normally both normal stresses and shear stresses must be known, this disadvantage of the proposed approach is not considered significant.

The normal stress distribution required in the proposed formula should be based on a bending moment that acts in the same plane as the applied shear force. However, this bending moment does not have to be the actual bending moment; any fictitious bending moment that acts in the plane of the shear force can be used.

By means of examples, the versatility of the proposed formula and its easy use in practical situations have been demonstrated.

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