On the concept of the virtual structure to provide an inextricable link between upper bound calculations and the virtual power principle

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Abstract On encountering the upper bound theorem of plasticity, students tend to be both fascinated and slightly mystified. They are favourably impressed by both the mathematical elegance of, and the significance of the output from, the application of the theorem, but they also perceive the origins of the theorem to be shrouded in mystery. Specifically, student confusion arises over the received wisdom that while virtual work arguments can be used to elegantly prove the upper bound theorem, upper bound calculations cannot, strictly speaking, be generally regarded as virtual work calculations on the original structure. As a result, the potential for innovation in structural analysis and design which stems from a firm grounding in the theorem may remain untapped in the students when they enter the design office upon graduation. The present paper tries to resolve this student enigma by arguing that if the concept of a virtual structure with special properties is introduced, then upper bound calculations on several frames can be shown to be unmitigated applications of the virtual work principle. It is then shown that reference to virtual power is more appropriate than that to virtual work, and that the hinges used in upper bound calculations are not meant to represent real plastic hinges, but rather are mathematical abstractions required for the work calculations; indeed, it is shown that the concept of the hinge applies with equal rigour to both plastic and elastic analyses. Finally a graph employing the idea of gradually varying section topology is presented to help students gain an intuitive feel for the concept of the shape factor as used for plastic analysis of structural sections. It is concluded that these ideas may well enhance students' appreciation of some key concepts of plastic theory.

Keywords upper bound theorem; virtual power; virtual structure; shape factor

Nomenclature

w	Load per unit length
L	Length of beam and height of column
M_B	Moment at location B of frame
M_p	Plastic moment capacity
W	Magnitude of point load
λ, λ_c, λ_S	General, combined mechanism and sway mechanism load factors, respectively
θ	Hinge rotation

Introduction

Within the spectrum of tools available to the modern engineer for analysis and design of a wide range of structures, plastic theory ranks among the most powerful. One main benefit of the theory is that its use leads to economic structures, because the full material strengths are utilised. Another important advantage of the theory is that residual stresses, which arise from effects such as differential thermal distributions and differential settlements of the structure, need not be considered because they have no bearing on the outcome of the application of the theory. This is an important feature, since quantification of residual stresses can be difficult.

The practical value of plastic theory manifests itself via either the upper bound theorem or the lower bound theorem. The upper bound theorem is popular because it entails minimal and elegant mathematical input, and can thus be used to design complex structures without being labour intensive. The theorem really comes into its own when used in mechanical engineering applications such as determination of the source load required to drive a metal-forming process, or for estimating the load required to fail a test specimen in a laboratory experiment. In both these cases it is important that, if anything, an overestimate of the load is obtained: in the first case so that the forming process runs, and in the second case to ensure that the loading rig can fail the specimen and so permit the maximum useful data to be obtained on the specimen's behaviour. The beauty of the upper bound theorem in both these applications is that it predicts a load equal to or greater than that required to achieve the desired end. Hence only one possible plastic state need be considered in the analysis, even if the state considered does not match that which will occur in reality. This saves considerable effort and time, while simultaneously providing great rewards, at the design stage. The theorem can also be used for ultimate limit state design of steel frames in either a mechanical or a civil engineering context.

Other potential applications of the upper bound theorem in civil engineering include its use to establish the (blast) loading required to demolish a derelict structure. Also, it is now increasingly used to assess the load-carrying capacities of bridges and buildings which either may need to be upgraded owing to a requirement to carry increased load, or may be suspect owing to loss of material, through corrosion of steel, for example. This structural-assessment application has been triggered by the current economic climate, which has prompted a shift in attitude from outright replacement of existing suspect structures, to reliable assessment of the structures' load capability with a view to strengthening if the structures are shown to be deficient. Again, the power of the upper bound theorem in such applications, if used properly, derives from the minimal volume of analytical matter required to produce a result.

For the structural-assessment application, it is important to note that, since the upper bound theorem generally overestimates the strength of the structure, care must be taken to consider plastic states which minimise the strength estimate and so provide as close an approximation to the true strength of the structure as is possible. This choice of collapse state(s) can require careful thought. Probably owing to the requirement for such care, and given the fact that this structural-assessment application of the upper bound theorem is still in its infancy, there are few commercial computer packages which perform this task. Moreover, it is not apparent in the foreseeable future that the development of such packages will keep pace with the anticipated gain in momentum of this application of the theorem. Hence, over the next few years, new structural engineering graduates will need to draw on personal skill when carrying out such assessments.

It is thus clear that there is an increasingly urgent need for engineering students to develop a profound understanding of the origins of the upper bound theorem. However, experience suggests that students find certain key issues surrounding the theorem and its use a trifle elusive. This situation has provided the motivation for the present paper, which uses a derivation of the theorem to provide a comprehensive treatment of the problem areas. A simple loaded structure, specifically a rectangular portal frame under vertical and horizontal point loads, is employed in the discussions. The frame is chosen owing to its use in both mechanical and civil engineering applications. For this frame, the ensuing discourse tries to emphasise the following main points:

1 Although the application of the theorem employs a mechanism and a plastic moment layout which are consistent with each other, the derivation of the theorem uses a mechanism and a plastic moment distribution which are independent of each other.
2 Although the virtual work principle forms the basis of an elegant derivation of the theorem, the application of the theorem cannot be viewed as a virtual work calculation as long as the original structure is perceived to be present when incorrect plastic mechanisms are considered.
3 If the concept of a *virtual over-strength structure* is introduced when the theorem is applied to an incorrect plastic collapse mechanism, then the application of the theorem is indeed a virtual work calculation. This idea is very important because, along with point 2 above, it demonstrates that the virtual work principle underpins the upper bound calculations. In their natural progression through structural engineering degree courses, students become aware of the power of the virtual work principle as applied to truss deflection calculations and then as an efficient route to derivation of stiffness matrices in finite element analysis. By introducing the idea of the *virtual structure* as described here, the students also become aware that the upper bound theorem is in this case yet another powerful application of the virtual work principle. On graduation, the students are thus aware of the broad and powerful range of applications of the principle.
4 The term *virtual power* is more appropriate than *virtual work* in the context of the upper bound theorem.
5 The hinges used in upper bound calculations are virtual hinges, not real ones.
6 The ideas of the hinges and the virtual power principle also apply when the structure is elastic. This places the concepts in territory which is more familiar to the students by the time they first encounter plastic theory, and helps emphasise the idea of virtual power as a fundamental technique which applies across the broad spectrum of structural engineering analytical and design activity.

The paper then deals with another valuable concept in plastic theory, namely that of the shape factor. Physical arguments are developed to provide a framework of ideas within which students may gain an intuitive feel for this concept and, more importantly, its variation with section topology. From these arguments emerges the revealing idea of representing variation of section topology as the abscissa of a shape factor graph.

It is concluded that the above ideas will assist in enhancing students' understanding of some fundamental and very useful concepts of plastic theory.

The upper bound theorem and the idea of a virtual structure

This section begins with a statement and derivation of the upper bound theorem. The derivation then naturally leads on to the concept of the virtual structure. Statements of the theorem are given in texts by Calladine [1], Horne [2], Heyman [3], Neal [4], Moy [5] and others. One form of the statement particularly pertinent to the present discourse is as follows:

If, for an infinitesimal movement of an assumed mechanism, an estimate of the plastic collapse load factor of a frame is made by equating the work done by the external loads to the work done against the hinge moments, then the estimate will be either high or correct.

Consider the rectangular portal frame and loading of Fig. 1, for which the potential collapse mechanisms are shown in Fig. 2. The frame is of uniform plastic moment capacity M_p. The combined mechanism of Fig. 2(c), with a collapse load

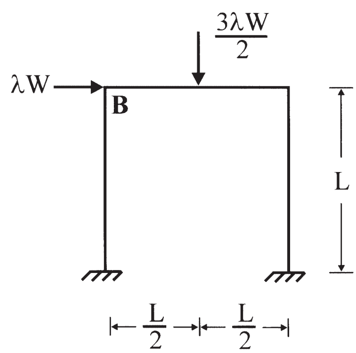


Fig. 1 *Frame and loading.*

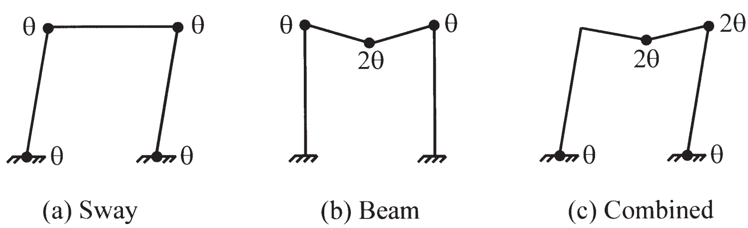


Fig. 2 *Potential collapse mechanisms.*

factor, λ_c, of 24M$_p$/7WL, is the correct collapse mechanism, and so the associated moment distribution satisfies both equilibrium and the yield criterion. The virtual work principle may thus be applied to a system comprising the loads and moment distribution associated with Fig. 2(c) as the equilibrium set, alongside rotations consistent with *any other* layout of hinges constituting a collapse mechanism, such as that of either Fig. 2(a) or Fig. 2(b), as the compatible set. If the rotations of Fig. 2(a) are chosen, virtual work gives:

$$\lambda_c \mathrm{WL}\theta = 3\mathrm{M}_p\theta + \mathrm{M}_B\theta \tag{1}$$

where λ_c is the correct collapse load factor (for the combined mechanism of Fig. 2(c)) and M_B is the moment, associated with λ_c, at location B of the frame (Fig. 1). The moment at B in the combined mechanism cannot violate the yield criterion, and so the following holds:

$$\mathrm{M}_B \leq \mathrm{M}_p \tag{2}$$

If we now perform an upper bound calculation for the sway mechanism of Fig. 2(a) in the usual manner, the following expression results:

$$\lambda_s \mathrm{WL}\theta = 4\mathrm{M}_p\theta \tag{3}$$

where λ_s is the collapse load factor associated with the sway mechanism of Fig. 2(a) in the upper bound sense. By combining equations (1) to (3), the following inequality emerges:

$$\lambda_s \geq \lambda_c \tag{4}$$

Equation (4) methematically expresses the upper bound theorem. The crux of the proof lies in the work calculations at the hinge of the sway mechanism which does not also exist in the combined mechanism. This hinge, at point B in Fig. 2(a), is henceforth termed the non-coincident hinge. At this hinge, the moment assumed for the sway mechanism in the upper bound calculation is greater than the (equilibrium) moment from the combined mechanism in the virtual work expression. At all other hinges of the sway mechanism, the moments (and hence the associated work expressions) are the same whether the upper bound or virtual work calculation is performed. Since the rotations are the same whether equation (1) or (3) is considered (because the same sway mechanism is considered in both cases), the inequality of equation (4) arises purely because of the difference in moments between the sway and combined mechanisms at the non-coincident hinge. If the combined mechanism and beam mechanism are considered, similar arguments apply. Indeed, the arguments apply to any number of hinges which are non-coincident between the correct mechanism and any incorrect mechanism of any framed structure under any type of load. The above result is thus general.

Since equation (3) is not obtained via multiplication of equation (1) by a constant factor, and since equation (1) is a direct application of the virtual work principle, it is deduced that equation (3) is not an expression of virtual work. Therefore, the plastic moment layout associated with the sway mechanism cannot satisfy equilibrium and also be applicable to the original frame of Fig. 1. This fact can alterna-

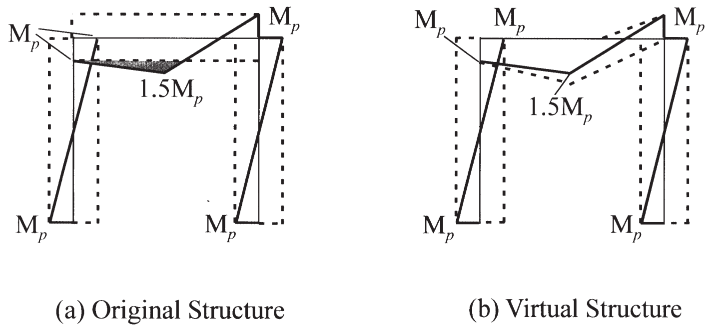


Fig. 3 *Equilibrium moment distribution (solid lines) and moment strength envelopes (dashed lines). Shaded region of (a) indicates violation of yield criterion.*

tively be demonstrated via a *reductio ad absurdum*, which also has the benefit that it naturally leads on to the concept of the virtual structure. We start by assuming that the equilibrium moment distribution that is consistent with the sway mechanism's upper bound collapse load factor (λ_s) of $4M_p/WL$ does not violate the yield criterion anywhere along the frame. Fig. 3(a) shows this equilibrium distribution, along with the moment strength envelope for the entire frame. (It is easily verified that this moment distribution satisfies equilibrium everywhere along the frame.) It is seen that the plastic moment capacity (M_p) *is* exceeded over much of the length of the beam (shaded region of Fig. 3(a)). Thus the yield criterion *is* violated, and so the original assumption (that this moment distribution can exist in the frame) is absurd.

Note that when the upper bound calculation is applied to the combined mechanism of Fig. 2(c), both equilibrium and the yield condition are found to be satisfied. Hence the upper bound calculation is in this case a direct application of the virtual work principle.

From the above discussion, the following important points emerge:

1 If the view is taken that upper bound calculations are always performed for the original structure, then in general upper bound calculations are not direct applications of the virtual work principle, unless the mechanism considered is the correct collapse mechanism.

2 Notwithstanding the above argument, an elegant proof of the upper bound theorem is obtained via application of the virtual work principle. The proof uses the hinge rotations from the incorrect mechanism under consideration and the moment distribution from the correct mechanism. The route to the proof of the theorem thus contrasts with the application of the theorem, where the plastic moment locations coincide with the hinge locations.

Let us now continue the discourse initiated via the *reductio ad absurdum*. We do this, for the moment distribution of Fig. 3(a), by considering a fictitious structure, or in other words a virtual structure, that is the same as the original structure except that the moment strength envelope for the virtual structure is as shown in Fig. 3(b).

The strength envelope of Fig. 3(b) is seen to lie outside the equilibrium distribution everywhere except at the hinges of the sway mechanism, where the strength envelope and equilibrium distribution coincide. Therefore, both the equilibrium and yield criteria are satisfied along the entire virtual structure. Since all three criteria of mechanism, equilibrium and the yield condition are satisfied, the sway mechanism and associated moment diagram constitute the correct plastic collapse state for the virtual structure. We may thus confidently state that, for the virtual structure, the upper bound calculation is indeed an application of the virtual work principle.

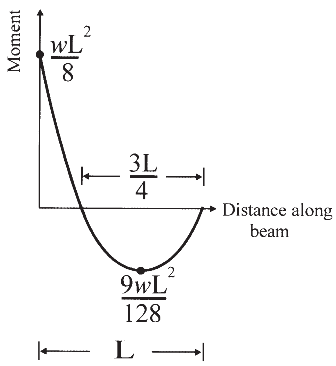
The strength envelope given in Fig. 3(b) is one of an infinite number that will coincide with the equilibrium distribution at the hinges and exceed it everywhere else. In other words, an infinite number of virtual structures exist for the sway mechanism. The same argument can be applied if we consider the (incorrect) beam mechanism of Fig. 2(b). Indeed, as long as the introduced plastic moment layout permits construction, throughout the entire structure, of a moment distribution in equilibrium with the external loads, then an infinite number of virtual frames exist which satisfy the equilibrium and yield conditions. In such cases, upper bound calculations are indeed direct applications of the virtual work principle. This leads to the following alternative statement of the upper bound theorem:

If, for an infinitesimal movement of an assumed mechanism, a plastic collapse load factor is calculated via equating the work done by the external loads to the work done against the hinge moments, the load factor will be correct for a virtual structure of strength at least equal to that of the original structure, and this calculation is a direct application of the virtual work principle to the virtual structure under the factored loads.

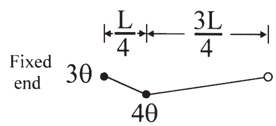
One main strength of this statement stems from its expression of the upper bound theorem as yet another powerful application of the virtual work principle.

Understand that upper bound calculations employ virtual hinges

An important point commonly misunderstood by students is that the hinges used in upper bound calculations are pure mathematical abstractions, employed solely for mathematical convenience and not for reasons of representing the mechanism in the real structure at incipient collapse. The hinges are used because they correspond in a work sense to the plastic moments, and because they ensure that only these (plastic) moments enter the work expression. Hence the resemblance of the upper bound hinge mechanism to an 'actual' collapse mechanism arises purely from the fact that the geometry of the 'actual' collapse mechanism is also dictated by the locations of the plastic moments. Outside of that link through the plastic moments, the upper bound hinges constitute different beasts from the 'rotations' associated with the failure behavior of the frame. Indeed, note that since each hinge rotation in an upper bound calculation occurs over zero length, infinite strain capacity is required of the frame at each hinge. In reality, by contrast, the strain capacities of structural materials (even mild steel) are finite, and so full hinges do not occur. In addition, note that the moment – curvature plot for a section of perfectly plastic material approaches the plastic moment asymptotically, and so the full plastic moment does not occur until infinite curvature develops.



(a) Moment diagram



(b) Mechanism

Fig. 4 *Elastic analysis of a propped cantilever.*

The idea that the hinges are fictitious can be reinforced in the students' minds by reminding them that mechanism-based, and hence hinge-based, calculations are equally valid when applied to fully elastic structures, where it is readily seen that no plastic hinges exist. To that end, consider a propped cantilever of span L, of constant section properties, and under a uniformly distributed load of *w* per unit length. The elastic moment diagram for this scenario is given in Fig. 4(a). Let us use this moment distribution as an equilibrium set and the mechanism of Fig. 4(b), with hinges at the fixed end and at the point of contraflexure, as a compatible displacement field. The work associated with the moments at the hinges is zero for the contraflexure hinge and $3wL^2\theta/8$ at the fixed-end hinge, giving a total of $3wL^2\theta/8$. If the work done by the external loads is determined for this infinitesimal movement of the mechanism, this is also found to be $3wL^2\theta/8$. Hence the external and internal work done equate, consistent with the virtual work principle, but there are no real hinges involved when elastic behaviour is considered. The mechanism used is thus a virtual one, based on a series of virtual hinges. The virtual work principle holds because the moments used are in equilibrium with the external loads, and applies with equal rigour irrespective of the regime of behaviour (elastic or plastic) of the structure. The argument always comes round to the idea that virtual work is the fundamental principle common to both elastic and plastic analyses.

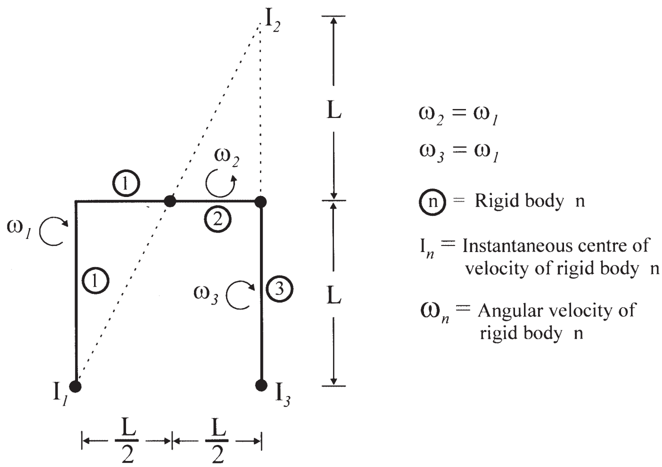


Fig. 5 *Full dynamic movement – instantaneous centres of velocity.*

To reinforce this idea that virtual hinge-based calculations do have powerful applications in elastic analyses, students' attention may be drawn to the Müller-Breslau principle, which introduces a virtual hinge at a point in order that the moment influence line for that point can be obtained from the resulting virtual deflected shape. The statement and application of that principle can be found in various structural engineering texts, such as those by Bhatt [6] and by Coates *et al.* [7].

Virtual work or virtual power?

In applying the virtual work principle to the sway mechanism of Fig. 2(a), we consider an infinitesimal quasi-static movement of the mechanism from the original position of the frame. In so doing, the horizontal distance moved by the point of application of the horizontal load (point B of Fig. 1) is taken as $L\theta$. In fact, the horizontal distance moved by point B is $L\sin\theta$, so strictly speaking the expression $L\theta$ is an approximation, albeit a good one which gets better as the angle θ is reduced. In addition, the beam moves a vertical distance $L(1 - \cos\theta)$ during this infinitesimal sway of the frame, so work of magnitude $3\lambda_s WL(1 - \cos\theta)/2$ is done by the vertical load at the midspan of the beam. It is only when we consider the limit of the approximation between $\sin\theta$ and θ, and also between $\cos\theta$ and 1, that we end up with the final expression for the external work as $\lambda_S WL\theta$.

This need for introducing approximations can be eliminated if, from the start, the movement of the sway mechanism is considered to be fully dynamic rather than quasi-static. Under dynamic conditions, the frame may be considered as a combination of rigid bodies, with each body having its own instantaneous centre of velocity, as shown in Fig. 5. Once this is done, the external loads simply need to be multiplied by the corresponding linear velocities, and the moments by the corresponding relative angular velocities at the hinges, to give an expression which relates to conditions only at the instant at which the mechanism is considered. Hence no approximations (which would have to be ‘tidied up’ by considering what happens as movements approach zero) need be considered – the expressions are exact from the start. However, in taking the product of force and linear velocity, as well as of moment and relative angular velocity, what we have achieved is an expression giving ‘virtual work per unit time’ or, in other words, ‘virtual power’. Coupled with the arguments given in the preceding section, this means that *upper bound calculations are applications of the virtual power principle to the virtual structures defined above*. This idea of virtual power is consistent with the approaches of many texts on plasticity.

Useful intuitive ideas on the concept of the shape factor

The shape factor of a structural (commonly steel) section is defined as the ratio of the section’s plastic moment to its initial yield moment. This is a useful parameter, in that it gives an idea of the reserve of moment capacity available in the section beyond the moment at which yield begins at the extreme fibre(s) of the section. Students are generally happy with the method of calculating shape factors. The present section aims to complement this mathematical competence with an intuitive feel for the idea of the shape factor as a geometric concept, without recourse to calculation. The discussion given here focuses on sections with two axes of symmetry.

The ideas which underpin this discussion are simple. For convenience, *intermediate material* is defined as the material between the extreme fibre(s) and the neutral axis of the section under consideration. Plasticity of a structural section in flexure starts at the extreme fibre(s) and then progresses incrementally through the intermediate material towards the neutral axis. If the intermediate material is far away from the extreme fibres, then the additional moment which develops as this progressive plastification takes place will be large. However, if most of the intermediate material is near to the extreme fibres, then plastification of this intermediate material generates little additional moment beyond that at the onset of section yield. Hence, as the intermediate material is progressively redistributed such that less and less of it is located near the extreme fibres, the additional moment capacity introduced after onset of yield increases, and so the shape factor increases. Conversely, increasing the concentration of intermediate material near the extreme fibres will lead to a progressive decrease in shape factor. At the limit, as the intermediate material is bunched near the extreme fibres, with vanishingly small quantities left elsewhere (i.e. towards the neutral axis), the shape factor asymptotically approaches a value of 1. Since the distribution of intermediate material defines the shape, or topology, of the section, it is easy to see why, from the above arguments, the nomenclature ‘shape factor’ is adopted: the parameter is indeed a function only of the shape of the section.

The above arguments suggest that, by incrementally varying the distribution of intermediate material between the extreme fibres and the neutral axis, a progressive variation of section topology may be represented. Hence a graph illustrating the

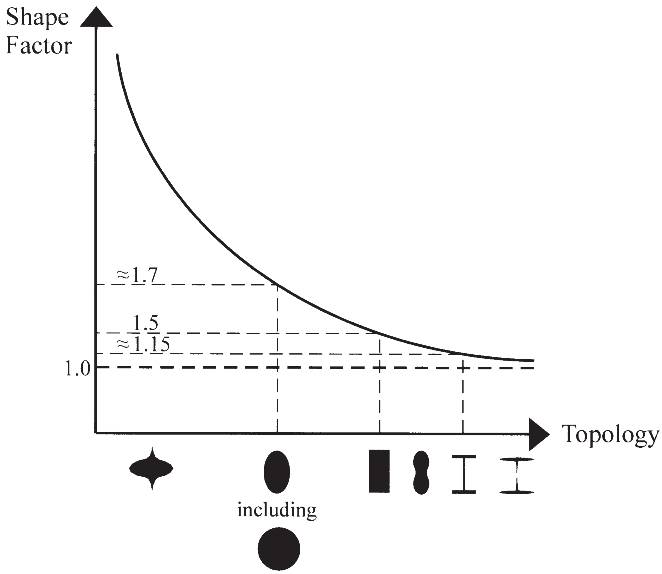


Fig. 6 *Shape factor versus section topology.*

variation of shape factor with topology can be drawn. Such a graph is shown in Fig. 6, with topology represented as the abscissa and shape factor as the ordinate. In proceeding from left to right across the abscissa axis, the topology changes incrementally by progressive squeezing out of intermediate material from the vicinity of the neutral axis and by accumulation of this squeezed material near the extreme fibres of the section. Note the presence of familiar shapes such as the ellipse, the circle and the I-section. The shape factor of the ellipse is constant irrespective of the axis (major or minor) of bending. The coincidence of the ellipse and the circle along the abscissa axis is a direct consequence of the fact that these two shapes, like isoparametric finite elements, are members of the same topological family.

A note of caution. On encountering Fig. 6, students are initially bemused by the representation of topology as a non-numerical concept along the abscissa axis. These doubts are easily eliminated once it is explained that *any* effect which can be shown to change incrementally can be represented along a line. The point is usually effectively driven home when it is explained that counting through integers 1, 2, 3, 4 . . . represents just one example of such an incremental change; gradual variation of section topology, as illustrated in Fig. 6, is yet another.

Fig. 6 is useful in that it permits a rapid rough evaluation to be made of the shape factor of any new doubly symmetric structural section which may be encountered. If the section does not match one of those given in Fig. 6, some form of ‘geometric interpolation' between the two nearest shapes on Fig. 6 can be performed. Such rapid evaluation is useful when the section concerned is not compact: the rapid evaluation gives a correspondingly quick indication of the economy which may be achieved by using stiffeners to take the section up to its full plastic moment capacity.

Conclusions

A rectangular portal frame under vertical and horizontal loads has been used to illustrate some interesting ideas on the upper bound theorem of plasticity as applied to frames. In addition, some simple ideas, based on the progressive change of shape of doubly-symmetric structural sections, have been used to try and give an intuitive physical feel for the concept of shape factor. From the discussions, the following conclusions may be drawn:

1 For several loaded frames, it may be shown that if a well defined virtual structure is introduced during upper bound calculations, such calculations then constitute direct applications of the virtual work principle, even when the incorrect mechanism is under consideration. This requires, first, that an equilibrium moment diagram be constructed consistent with the upper bound load factor for the mechanism. The virtual structure has a moment strength envelope which coincides with this equilibrium moment distribution at the hinges of the mechanism and exceeds this equilibrium distribution everywhere else.
2 The above idea is useful, as it gets students to see that the utility of the virtual work method is not confined to elastic and finite element applications, but also extends to plastic analysis.
3 Students should appreciate that the hinges used in upper bound calculations are virtual, not real. Indeed, these mathematical hinges require infinite strain capacity, which is not available in real materials. This erroneous idea of real hinges may be dispelled by demonstrating that hinge-based calculations can be applied with equal success in elastic analysis, where the students are happy with the idea that no real hinges exist. To reinforce that point, the Müller-Breslau principle has been cited as an example of a hinge-based elastic analytical tool.
4 When quasi-static displacements of the mechanisms are considered during upper bound calculations, there are inherent approximations which must be eliminated using limit arguments. If, instead, dynamic movement is considered, the expressions are exact from the start. In such cases 'power' is a more fitting terminology than 'work', so the term *virtual power* is preferential to *virtual work*.
5 The progressive migration of material from the neutral axis to the extreme fibres of a structural section defines an incremental change of shape (or topology) of the section. This section topology change may be non-numerically represented as the abscissa of a plot of shape factor versus topology. The resulting graph assists with the development of an intuitive physical feel for the concept of the shape factor, and is useful for making quick estimates of shape factor for new section shapes encountered.

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