Study of second regions of dynamic instability of a uniform beam

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Abstract Dynamic instability of a beam is solved for second regions of instability. The governing equations are of the Mathieu–Hill type, where the parametric regions are sought. While the solution to the first region of a beam has been widely reported, there is none available in the literature for the second region. In this work, the regions are determined by the finite element method. Bolotin determined the solution analytically, but it is more of a qualitative solution of the problem. In this work, a quantitative solution is sought from the finite element method. The width of the regions are significant. The effects of the static load factor and the dynamic load factor are obtained.

Keywords second region; dynamic instability; beam; finite elements; bolotin

Introduction

The first mathematical analysis of a problem in dynamic stability was performed by N. M. Beliaev [1] in 1924. This problem was also solved by Bolotin [2]. He developed solutions to both the linear and non-linear dynamic stability of bars, as well as plate problems. As in the case of the applied theory of vibration, Bolotin [2] did not include the inertia forces associated with the rotation of the cross-sections of the rod with respect to its own principal axes. The finite element method for the study of dynamic stability behavior was first used by Brown *et al*. [3] in studying a Euler-Bernoulli beam. Timoshenko and Gere [4] studied the effect of shear deformation on the static buckling load. Thomas and Abbas [5] considered the effect of shear deformation and rotary inertia on dynamic stability. Datta and Chakraborty [6] have studied the behavior of parametric instability of tapered beams by the finite element method. Datta and Nagraj [7] also studied the dynamic instability behavior of tapered bars with flaws supported on an elastic medium.

The common point that emerges from this literature is that the basic governing equation is of the Mathieu–Hill type [3]. The characteristic equation can be formed by considering two linearly independent solutions and reducing the equation to diagonal form with the help of initial conditions. It is also shown that the dynamic instability of a column is a form of Mathieu–Hill type of instability, with solutions having period T and $2T$. All of the available literature deals with the first region of instability, $2T$, and neglects the second region of instability, with period T.

This paper deals with the determination of these neglected second regions of instability by the finite element method. It is the first work in the literature to address these second regions of instability. In this paper, a finite element method is used to address this problem. Bolotin [2] obtained a qualitative solution to this problem. The

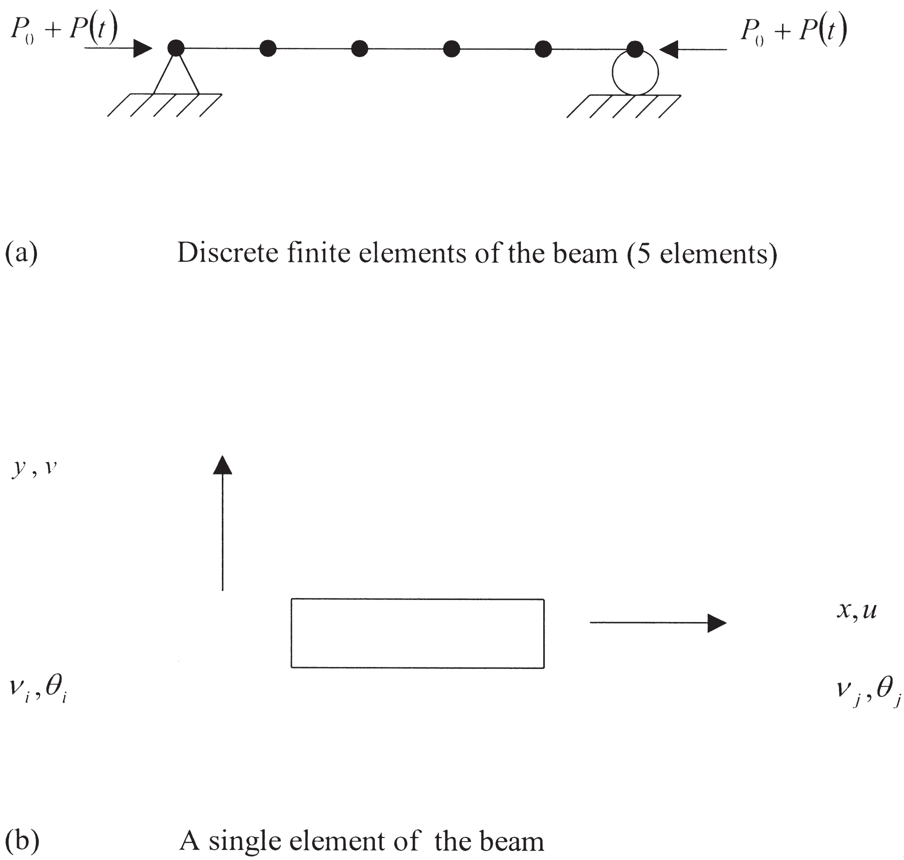

Fig. 1 *Simply supported uniform beam.*

present paper obtains a quantitative solution. Further, this work is the first in the literature to obtain the neglected second regions of instability of a beam.

Mathematical formulation of the problem

A simply supported bar, as shown in Fig. 1(a), can be represented by an assembly of finite elements connected together at the nodes. A typical finite element, as shown in Fig. 1(b), has v_i, θ_i and v_j, θ_j as the generalized coordinates. The matrix equation for the free flexural vibration of an axially loaded discrete system in which rotary inertia and longitudinal inertia are neglected is,

$$[M]\{\ddot{q}\} + [K_e]\{q\} - [S]\{q\} = 0 \tag{1}$$

in which $\{q\}$ = generalized coordinates, $[M]$ = mass matrix, $[K_e]$ = elastic stiffness matrix and $[S]$ = stability matrix, which is a function of the axial load, P. It should be noted here that the axial inertia term in this equation is not included. This system is subjected to a periodic force, $P = P_0 + P_t \cos\Omega t$, where Ω is the disturbing frequency and in which $P_0 = \alpha P^*$ and $P_t = \beta P^* \cos\Omega t$, with α and β as percentages of the static buckling load P^*. Thus the above equation is transformed into:

$$[M]\{\ddot{q}\} + \left([K_e] - P^*\alpha[S_s] - P^*\beta\cos(\Omega t)[S_t]\right)\{q\} = 0 \qquad (2)$$

If the static and the time-dependent loads are applied in the same manner, then:

$$[S_s] \equiv [S_t] = [S] \qquad (3)$$

As discussed by Brown *et al*. [3], this is a Mathieu–Hill type of equation, with period T and $2T$. The solution with period $2T$ (first regions of instability) is,

$$\{q\} = \sum_{k=1,3,5}^{\infty} \{a\}_k \sin\left(\frac{k\Omega_t}{w}\right) + \{b\}_k \cos\left(\frac{k\Omega_t}{2}\right) \qquad (4)$$

However, for a solution with period T (second regions of instability) it is,

$$\{q\} = b_0/2 + \sum_{k=2,4,6}^{\infty} \{a\}_k \sin\left(\frac{k\Omega_t}{2}\right) + \{b\}_k \cos\left(\frac{k\Omega_t}{2}\right) \qquad (5)$$

Equation (5) is the solution procedure which is covered in this paper. This equation is substituted in equation (2) to obtain the solution of the governing equation of motion. When that procedure is carried out and the simplification steps are performed, corresponding to the first two terms of the coefficient $\sin(\frac{k\Omega_t}{2})$ and $\cos(\frac{k\Omega_t}{2})$, the equations are obtained in the form of a determinant with matrix coefficients. The resulting solution is similar to solving a modal eigenvalue problem, which is available in any standard textbook [e.g. 8]. The only difference is the presence of axial load with a static component and a dynamic component with disturbing frequency Ω. The boundaries of the regions of instability, which are given by solutions with period T, are then expressed by:

$$\left(\begin{vmatrix} [[K_e] - P^*\alpha[S]] & [-P^*\beta[S]/2] \\ [-P^*\beta[S]/2] & [[K_e] - P^*\alpha[S]] \end{vmatrix} - \Omega^2/4 \begin{vmatrix} [M] & [0] \\ [0] & 4[M] \end{vmatrix}\right)\{q\} = 0 \qquad (6a)$$

And the other equation to bound the stability region is,

$$\left(\begin{vmatrix} [[K_e] - P^*\alpha[S]] & [-P^*\beta[S]/2] \\ [-P^*\beta[S]/2] & [M] \end{vmatrix} - \Omega^2/4 \begin{vmatrix} [0] & [0] \\ [0] & [M] \end{vmatrix}\right)\{q\} = 0 \qquad (6b)$$

Equations (6a) and (6b) can include the periodic terms if the eigenvalues are obtained in terms of the disturbing frequency, Ω. They define the boundaries of the region and should be compared with the boundaries of the first regions of dynamic stability, with period $2T$ [5–7], for which the assumed solution was shown in equation (4). Corresponding to the solution of period $2T$ (the first regions of dynamic instability), the stability boundaries are obtained as,

$$\left([K_e] - P^*\alpha[S] \pm \frac{1}{2}\left[-P^*\beta[S]\right] - \Omega^2/4[M]\right)\{q\} = 0 \tag{7}$$

At this stage it is important to consider the simplified case which is implicitly included in equations (6) and (7). In case of free vibration, $\alpha = 0$ and $\beta = 0$. Then, from both equations (6) and (7), neglecting the trivial solutions, we obtain the universally well known free vibration problem,

$$([K_e] - \lambda[M])\{q\} = 0 \tag{8}$$

Here, $\lambda = \mathrm{k}\Omega$, where k is a constant. To determine the regions of dynamic instability, the disturbing frequency, Ω, is taken as $\Omega = (\Omega/\omega_1)\omega_1$, where ω_1 = the fundamental natural frequency, as obtained from the solution of equation (8). This formulation is also used elsewhere [6].

Equations (6a) and (6b) are eigenvalue problems for second regions of instability of the form,

$$\left|[G] - \mu[H]\right| = 0 \tag{9}$$

The solution of this standard eigenvalue problem gives the mode shapes and modal frequencies, which are useful for plotting the regions of instability. The eigenvalues of equations (6a) and (6b) bound the regions. The matter of concern in solving eigenvalue problems of this type is whether G and H are symmetric, positive definite, singular, well defined, etc. It is to be noted here that not only unsymmetric components are present, but also the sparse-symmetric matrix H is singular and non-negative definite. This problem is solved by the 'GVCRG' subroutine in the IMSL (International Mathematical Systems Library) package available in the Cyber 180 platform at IIT Kharagpur. The programming language used was Fortran. The following properties of the beam were taken for numerical computations: length of the beam = 1 m; cross-sectional dimension = 2 × 2 cm; material density of the aluminum beam = 2800 kg/m^3; $E = 70 \times 10^9\, N/m^2$.

Solution by the finite element method

Equations (6a) and (6b) are solved in a suitable manner with the finite element method. It is required to have the stiffness matrix, [K], the geometric stability matrix, [S], and the mass matrix, [M]. These matrices have been derived by the authors in previous work [6, 7] and also in standard textbooks on the finite element method [9,

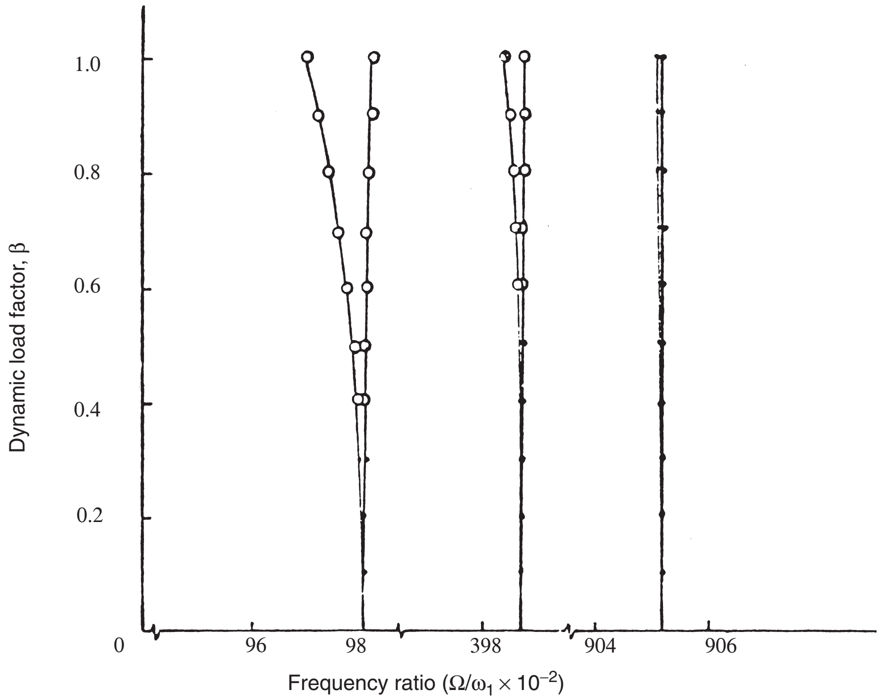

Fig. 2 *Second regions of dynamic instability for* $\alpha = 0.2$.

10]. When deriving these matrices, attention is restricted to flexural behavior of the beam [10]. Rotary and longitudinal inertia terms are neglected here. Here, μ is the mass per unit length of the beam. In this context, it should be pointed out that equations (6a) and (6b) are the boundaries of regions of instability. The shape of the stable and unstable boundaries for the first regions of parametric stabilities is well documented in the literature [3, 5–7]. However, this is the first work to derive the stability boundaries for second regions of instability by the finite element method in a quantitative approach.

Results and discussion

The results obtained for the static and dynamic behavior of a normal beam for various increasing static load factors are plotted in Figs 2–5. In each figure, the plot of increasing dynamic load factor versus the non-dimensional frequency ratio is shown. The regions of instability are bounded by the lines. The regions outside these bounds are stable. The plots show the first, second and third modes corresponding to each of the static load factors. It is seen that the width of the third and second

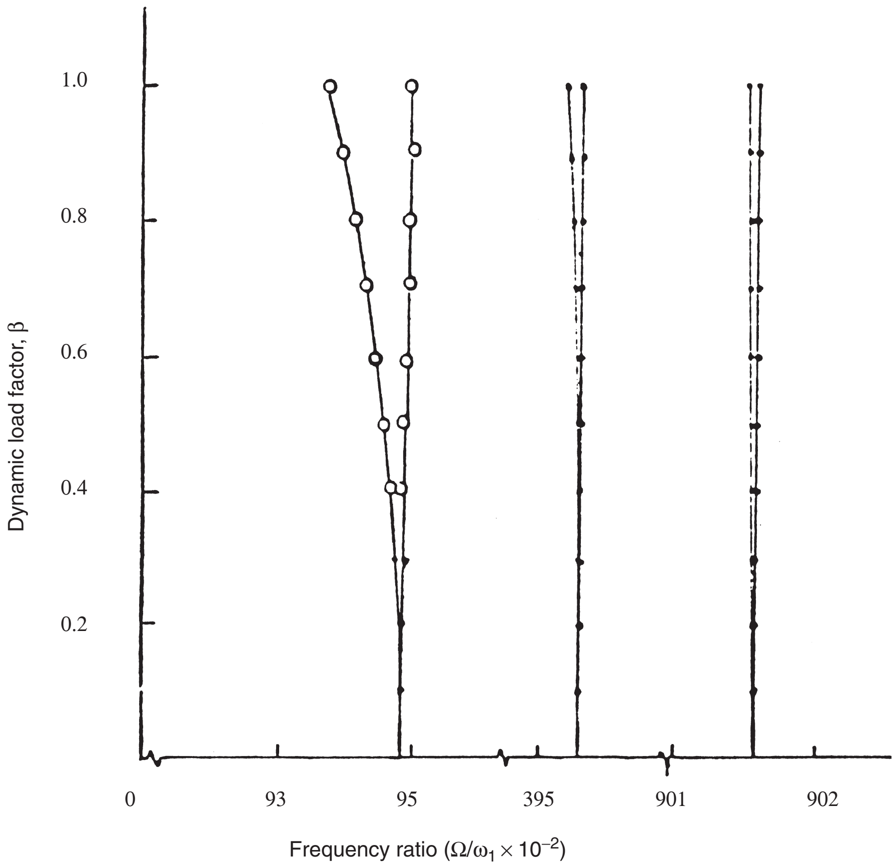

Fig. 3 *Second regions of dynamic instability for* $\alpha = 0.5$.

modal regions are much less than that of the first modal regions. The width of the unstable regions increases with increasing dynamic load factor. The effect of the static load factor is seen in the shift of the unstable regions towards a decreasing frequency ratio. In other words, as the static load increases, instabilities are obtained at a lower excitation frequency. This behavior is similar to the effects seen in the parametric instability regions for first regions of instability [5, 7]. In the present case, the *width* of the instability regions suggests that these second regions of instability *cannot be neglected*. A more detailed study of these instability regions is needed. It is hoped that this work will serve as a catalyst for the further study of these regions of instability.

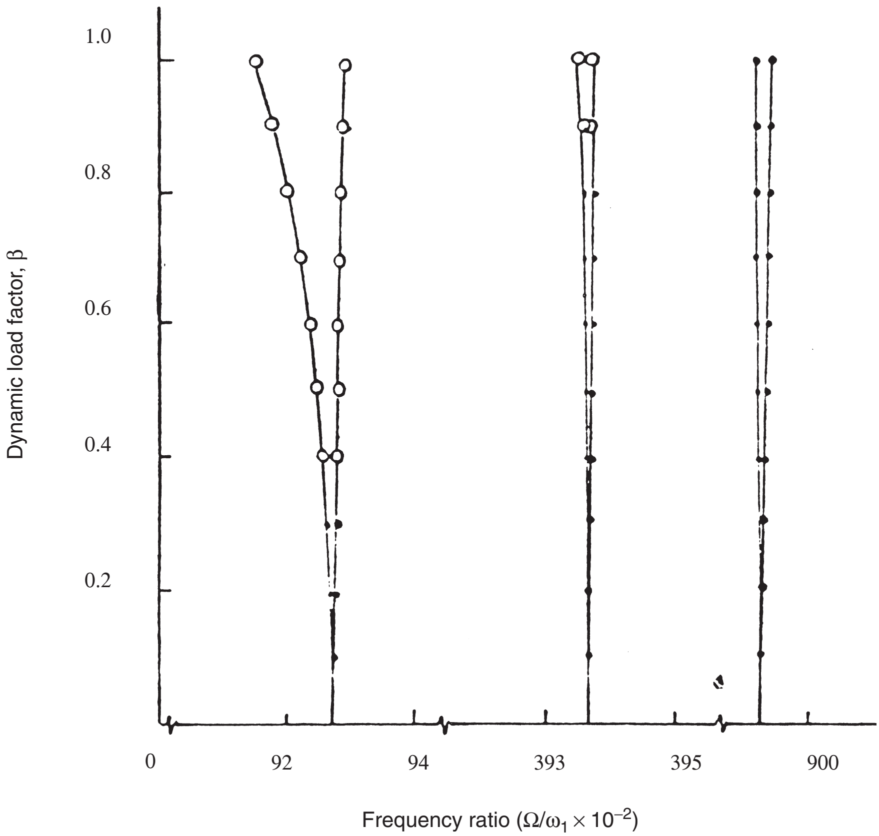

Fig. 4 *Second regions of dynamic instability for* $\alpha = 0.7$.

Conclusions

The neglected second regions of instability of a system must be taken into account in the proper design of a system. With an increasing dynamic load factor, the regions of instability increase in size. The effect of increasing the static load factor is to move the instability zones inward on the frequency ratio axis and to widen them, thus resulting in instabilities at lower frequency ratios. The effect of the second region of instabilities decreases at higher modes. The neglected second regions of instability have significant size and must be taken into account in the proper design of a system.

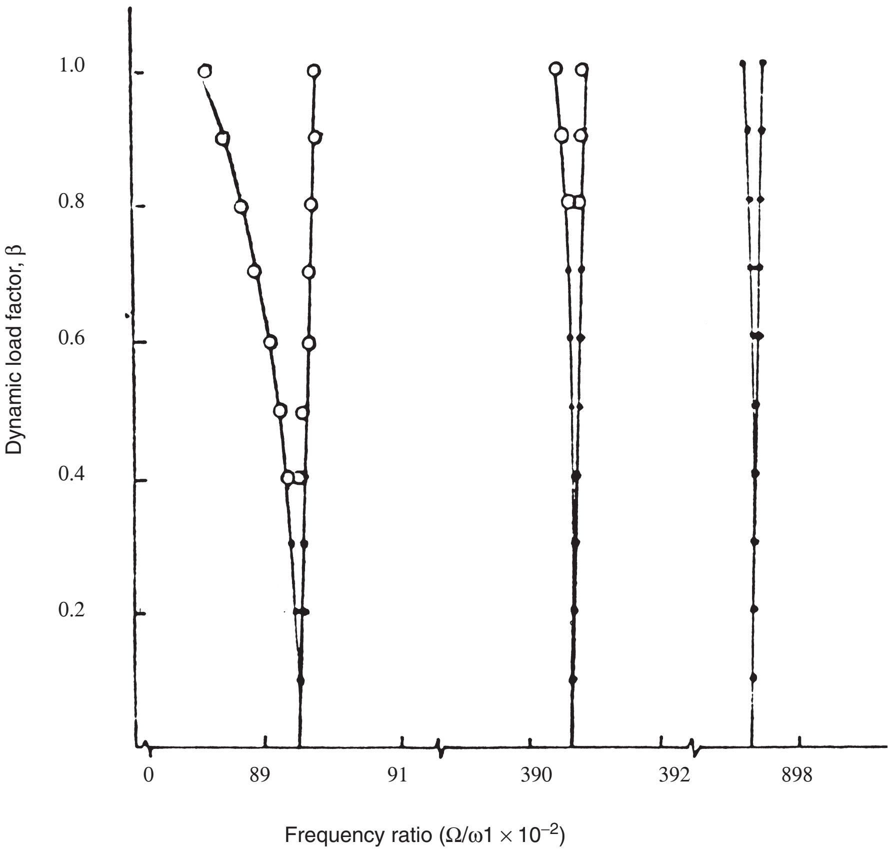

Fig. 5 *Second regions of dynamic instability for* α *= 1.0.*

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