Investigation of elastic constraint non-linearity

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Abstract This paper shows a solution for non-linear vibrations in a homogeneous elastic line consisting of three elements connected by non-linear elastic constraints. The solution obtained is a functional spectral series, each harmonic of which is determined analytically on the basis of a system of equations describing the vibration process when the same linear system is acted on by a series of forces. The process forces depend on the degree of elastic constraint non-linearity and on the vibration amplitude of the lowest-order harmonics. We show that the boundary frequency of each harmonic decreases in proportion to the order of harmonic, and the resonance spectrum of harmonics of the dynamic process contains a spectrum of natural frequencies lower than the natural boundary frequency and the spectrum of frequencies of lower-order harmonics located between the natural boundary frequency and the boundary frequency of the first harmonic. It is shown that the method of recurrent determination of the spectrum of a non-linear dynamic process can be extended to models with non-linear resistance and to the case of a complex-spectrum external force.

Keywords wave physics; mathematical physics; theoretical physics; many-body systems; non-linear dynamics; spectrum of non-linear dynamical process

Introduction

Vibrations surround us, from seismic vibrations of the earth and the internal oscillations of molecules to modern communications and transport, any mechanisms and constructions. Safety often depends to a large extent on the reliability of methods of calculation of vibration systems. These methods, mainly matrix, are known to be far from perfect, so most of these systems are calculated approximately and numerically. They require highly laborious processing, especially in relation to lumped systems and systems that cannot be reduced to the simplest basic models.

Most real-world models are conventionally modelled for calculations by non-linear systems; this is the most voluminous, complicated and laborious class of problem. As a Soviet *Physical Encyclopaedia* states [1]:

The circumstance that the non-linearity of general equations of the elasticity theory has a double nature causes the following classification of problems of this theory:

a) Linear problems in which the extensions, shears, rotation angles of separate elements are small in comparison with the unity, being the values of the same order. . . .

b) Geometrically non-linear but physically linear systems where the rotation angles of spatial elements greatly exceed the extensions and shears, and the values of these last allow use of Hook's law. . . .

c) Physically non-linear but geometrically linear problems where the extensions, shears and rotation angles are small in comparison with the unity and comparable in their values, but the conditions of Hook's law are violated. . . .

d) Geometrically and physically non-linear problems.

Serious problems in their solution make necessary such a detailed classification of dynamic problems. First of all, we still do not have a unified method to solve non-linear problems. These is rather a multitude of particular techniques, each of which, along with its advantages, brings to the calculation its own difficulties. It is fortunate if we know an explicit formula for a set of solutions (an aggregate of motions) [2, p. 12] and it is not often that we can obtain a direct solution of the problem. When choosing an indirect technique, researchers find that there still is no unified theory of vibrations of strongly non-linear systems that does not feature small parameters and 'strange' properties, even when considering quite simple model systems [3, p. 7]. The narrow applicability of conventional methods also is a hindrance. In particular:

applicability of the Krylov–Bogolyubov method is practically determined not by the approximations convergence when their number growing, but by the asymptotic properties of series with the fixed number of terms of a series and ε_r tending to zero. [4, p. 308]

This limits the applicability of indirect methods to a narrow edge area of non-linear mechanics [2, p. 12]. Additionally, the indirect methods give uncoordinated solutions with no idea of the structure of the set of solutions as a whole [2, p. 12].

It is still impossible to calculate the real frequencies of a non-linear system. The mathematical tool of the theory of series is cumbersome and allows the calculation of only a small number of terms; furthermore, it does not give a way to express the common term and the sum of these series [5, p. 305]. The resonance conditions and behaviour of the system either close to or far from resonance are also a great problem [5, p. 309].

All existing methods join the solutions for different parts of a system, based on the continuity conditions. In addition, the methods are highly sensitive to any attempts to vary some parameters of the system in the course of solution, which themselves give rise to the need to calculate anew. Besides, the apparent high accuracy of many methods is known to be often illusory [6, p. 317].

The approaches are limited by the matrix, integral and asymptotic methods. Together with the firmly established practice of giving additionally the initial and boundary conditions for a generalised system of differential equations, it brings an insurmountable problem:

the presence of irregular boundaries in the majority of practical problems disables the construction of the analytical solutions of differential equations, and the numerical techniques become the only possible means to obtain quite accurate and detailed results. [7, p. 12]

The numerical techniques, with all their simplicity and broad applicability, are completely useless when we need to process them logically or mathematically. Their great sensitivity to the chosen step is another demerit [8, p. 9], since two nearest points of the system can behave very differently, and we always risk the omission of an important feature of the system's behaviour. However, the main shortcoming of the numerical techniques is the absence of a reliable analytical formalism, which, if having been used as the basis of calculations, would factually predetermine the quality of the numerical solutions obtained:

> As a rule, the search of solutions was carried out by different techniques (Chesare, Krylov–Bogolyubov, through the variable action-angle etc.) for different cases, with the expansion of $\sin x$ and $\cos x$ into a series in the orders of x smallness. Such diversity of techniques has impeded the evaluation of particular solutions, the interpretation of obtained results and understanding of the reasons for chaos and bifurcation in the systems. [3, p. 36]

The analytical record of solutions enables us to avoid all these shortcomings. It is much more simple and accurate, gives good understanding of the process in local sections and, as a whole, is easily programmable, reduces the laborious calculations and saves operative machine memory [8, p. 9]. We will show in this paper why, because of the absence of exact analytical solutions for the whole complex of linear problems, the conventional solutions for non-linear problems were limited to the values of natural frequencies of the modelled system. This obviated study of the regularities of processes in a form achievable analytically. Here, we yield the solutions in an analytical form.

The problems discussed above led us to develop a new, non-matrix method of obtaining exact analytical solutions, which is able to solve a broad range of problems for one-dimensional elastic lines (such as mechanical shafts), for bound systems (such as seismic systems), for closed-loop systems (allowing, for example, modelling of the dynamic processes in a wheel) and for complex elastic systems containing resonance sub-systems, etc. We have also successfully applied our method to the calculation of mismatched electric filters and checked it experimentally for the electric circuit of six mismatched ladder filters (as this is beyond the ability of conventional calculations to check). The experimental plots obtained coincided well with those calculated.

Up to now our method had a great demerit – it was developed only for linear systems, when it is non-linear systems that are the most difficult to calculate. The aim of this paper is to fill this gap. We will make use of the advantages of exact analytical solutions [9–16] and extend these into the area of non-linear dynamics, neglecting the smallness of non-linearity of constraints. The method is not based on the linear solutions and the solution for the non-linear system is not obtained by varying the differential equations for the linear solution. Instead, we will try to transfer to the non-linear mechanics the basic principle grounded on the clear specification of the features of the model. This is because [9–16], if we completely account for the features of the specific model in the modelling system of differential equations, the additional initial and boundary conditions become excessive. Instead, the boundary conditions are reflected in the features of the very system of differential equations, and the initial conditions are determined by the pattern of external effects of the forced vibrations, either by the features of the vibrations or by the selected element in the case of free vibrations. With this method, it is unnecessary to join the solutions; it enables the maximal determinacy of solutions to be obtained for the specific elastic system modelled. In its turn, this enables us to pass to the non-linear dynamics of process while keeping the continuous analytical relation with the linear solutions.

Statement of the problem

In order to just concentrate on the technique to seek solutions to the non-linear problem, let us choose as a study model a rather simple one-dimensional homogeneous finite line with unfixed ends (see Fig. 1), consisting of only three elements, connected by the non-linear elastic constraints having the characteristic $s(\Delta)$, where Δ is the degree of constraint transformation. Given the inertialess constraints, suppose that Δ determines the total shift of both joint bodies for each constraint. We will see below that this assumption will slightly change the standard form of the differential modelling equations, and this will make the search for a solution easier.

Determining the pattern of the stiffness coefficient, dependent on the degree of constraint deformation, we will not diverge from the standard conception, according to which the non-linear characteristic of an elastic constraint can be presented as a power series in Δ [17, p. 327]:

$$s(\Delta) = s_1\Delta + s_2\Delta^2 + s_3\Delta^3 + \dots \tag{1}$$

Practically, in many cases two terms are sufficient:

$$s(\Delta) = s_1\Delta + s_3\Delta^3 \tag{2}$$

This is because ‘The cubic (not quadratic) term provides the similar value of the resetting force with positive and negative shifts’ [17, p. 327]. Although, as we will show further, the limited distribution (equation 2) does not have any effect on the approach itself to solving the problem (but only on the result conditioned by the correctly taken expansion coefficients in equation 2 in each specific problem), we will use the conventional practice and, when expanding, confine ourselves to the cubic term. At the same time, we will not introduce an additional requirement of the smallness for s_3 in comparison with s_1, because in the course of investigation we will reveal that such a condition, inherent in all asymptotic techniques, is excessive in the approach we use.

Noting the above preliminary consideration and the results of [3], we can present the modelling system of differential equations as:

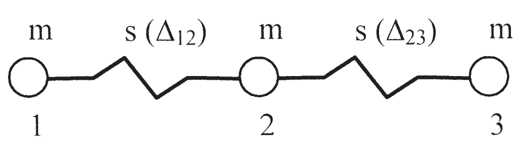

Fig. 1 *The mechanical model of a system of three-bodies (each mass* m*) connected by the non-linear elastic constraints whose stiffness coefficients depend on the total shift of the first and second bodies,* Δ_{12}*, and of the second and third bodies,* Δ_{23}*, correspondingly.*

$$
\begin{aligned}
&m\frac{\mathrm{d}^2\Delta_1}{\mathrm{d}t^2} = F(t) + s_1(\Delta_2 - \Delta_1) + s_3(\Delta_2 - \Delta_1)^3 \\
&m\frac{\mathrm{d}^2\Delta_2}{\mathrm{d}t^2} = s_1(\Delta_3 + \Delta_1 - 2\Delta_2) + s_3\left[(\Delta_3 - \Delta_2)^3 - (\Delta_2 - \Delta_1)^3\right] \\
&m\frac{\mathrm{d}^2\Delta_3}{\mathrm{d}t^2} = s_1(\Delta_2 - \Delta_3) + s_3(\Delta_2 - \Delta_3)^3
\end{aligned}
\tag{3}
$$

We also will not complicate the regularity of the external force and take it in a simple harmonic form,

$$
F(t) = F_0 e^{j\omega t} \tag{4}
$$

Finally, without limiting the generality, suppose that free vibrations are absent in the system considered and that, with settled motion, only non-damping forced vibrations are practically important [18, p. 58].

Given equation 4 and all factors completing the definition of the model, equation 3 gains complete determinacy and does not require any additional initial or boundary conditions. As we showed in [9–16] for different elastic linear models, the completely determined systems of differential equations do not need complete definition, as the constants that need determination on the basis of initial or boundary conditions are absent in their solutions. In the absence of free vibrations and with the settled pattern of the external force (equation 4), whichever pattern of the reaction of the line we obtain, the location of any element of the system will be uniquely determined by this reaction to the given external force. Given additionally the initial conditions with the chosen form of modelling equations, we would simply double the conditions that we can obtain, substituting the value of the initial moment of time to the solution obtained.

Thus, the form of record of the system (equation 3), in conjunction with equation 4, is fully determined with reference to the features of the elastic line studied, the external force and the initial conditions.

Solution-seeking technique

In seeking the solution for equation 3, note that if $s_3 = 0$ this system reduces automatically to a linear system, whose solutions we know [10]:

$$
\begin{aligned}
&\Delta_i = -\frac{F_0 \cos(7-2i)\pi}{\omega\sqrt{s_1 m}\,\sin 6\pi} e^{j\omega t}, \quad \beta < 1 \\
&\Delta_i = (-1)^i \frac{F_0\left(\gamma_+^{7-2_i} + \gamma_-^{7-2i}\right)}{\omega\sqrt{s_1 m}\left(\gamma_+^6 - \gamma_-^6\right)} e^{j\omega t}, \quad \beta > 1 \\
&\Delta_i = (-1)^i \frac{F_0(7-2_i)}{12 s_1} e^{j\omega t}, \quad \beta = 1
\end{aligned}
\tag{5}
$$

where $i = 1, 2, 3$;

$$\beta = \sqrt{\frac{\omega^2 m}{4s_1}}; \quad \tau = \arcsin\beta; \quad \gamma_+ = \sqrt{\beta^2 - 1} + \beta; \quad \gamma_- = \sqrt{\beta^2 - 1} - \beta \tag{6}$$

This makes it possible to seek exact analytical solutions for each separate harmonic in a simple way, whose features we will explain in the course of seeking the solution for each step (corresponding to the related harmonics).

In order to determine the harmonics step by step, let us draw our attention to the following feature. If we substitute, say, the periodical solution (at $\beta < 1$) from equation 5 into the general system (equation 3) at $s_3 \neq 0$, then, on the right part of each equality, an additional summand corresponding to the third harmonic will appear and violate the correspondence of equation 5 to equation 3. If we try to take into account the appearance of this additional harmonic, then in substitution the refined solutions to equation 3 there will appear the terms of the next, higher harmonic, etc. This corroborates the known fact that, due to the presence of non-linear terms, in the solution of equations of forced vibrations, the harmonics with frequencies equal to $n\omega_0$ will be inserted [4, p. 314].

This feature gives us the reason to seek the general solution as a series beginning with the fundamental harmonic corresponding to the external force frequency:

$$\Delta_i = \sum_{p=1}^{\infty} \delta_{ip} e^{jp\omega t} \tag{7}$$

where δ_{ip} is the (as yet unknown) momentary shift of the ith element of an elastic line (in this problem $i = 1, 2, 3$) corresponding to the pth harmonic of a non-linear dynamic process.

As we see, the absence of the condition of non-linearity smallness in the elastic constraints has led us to the essential change in the form of the solution sought. In particular, the parameter δ_{ip} in equation 7 has neither direct nor reciprocal power-type dependence, being typical for asymptotic techniques (see, e.g., [19, p. 45]), the same as parameter ε, indicating, for example, the smallness of function εQ (in comparison with the linear term) used in the Krylov–Bogolyubov technique [4, p. 314] in solving the problems of the kind:

$$\ddot{\xi} + \omega^2 \xi = \varepsilon Q\left(\xi, \dot{\xi}, \omega, t\right) \tag{8}$$

By its shape, equation 7 looks more like an expansion of a complex function into a Fourier series, which is usually inapplicable in solving non-linear mechanics problems by conventional methods. But with it the summation in equation 7 is carried out only for the positive values of p, and even the zero term is absent. Should we actually seek the solution in the form of Fourier expansion, we would not narrow the summation region without limiting the generality of the solution. However, as we will show below, the coefficients δ_{ip} are the resonance-type analytical functions that depend on the parameters of the elastic line and external force of frequency ω. And each of the coefficients in equation 7 will be the solution of its own system of algebraic equations; therefore, it will have its own functional dependence. At the same time, we know (see, e.g., [20, p. 214] or [21, p. 143]) that in the expansion of a Fourier series:

$$f(\tau)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}\left(a_k\cos kt+b_k\sin kt\right)\equiv\sum_{k=-\infty}^{\infty}c_k e^{jkt} \tag{9}$$

the coefficients a_k and b_k are real numbers, and the coefficients c_k are complex numbers that are determined from the equality:

$$c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\tau)e^{jk\tau}\mathrm{d}\tau \tag{10}$$

In other words, the form of solution for equation 7 differs from a Fourier expansion, so that it is a functional series, forming as a result of sequential compensation of higher harmonics that arise in the system of equations with the non-linearity. Proceeding from the fact that the expansion of stiffness coefficient (equation 1) contains only the increasing powers of Δ, the series-type solution beginning with the fundamental harmonic of the force applied (i.e., with $p \geq 1$) will describe the entire spectrum of harmonics arising in a non-linear elastic line as a result of the action of a pure harmonic force.

As a consequence of the above, the convergence of equation 7 will determine not the expansion Δ_i in harmonics $p\omega t$, as would take place in case of expansion Δ_i into a Fourier series; rather, it will indicate the existence of a finite solution in one or other region of the range. Thus the convergence of equation 7 will in fact determine the vibration stability in the range selected.

Proceeding from the above analysis, the technique to find the solution of a non-linear system of equations (3–4) can be determined as a sequential compensation of the residual terms of the higher harmonics when substituting the expansion of equation 7 into this system of equations. For it, we will substitute equation 7 into equation 3 and sequentially select the terms of corresponding harmonics, based on the fact that equation 3 can be zero only if the corresponding equations for all harmonics are zero.

To find the coefficients δ_{ip}, substitute equation 7 into equation 3, which will yield:

$$\begin{aligned}
&\sum_{p=1}^{\infty}p^2\omega^2 m\delta_{1p}e^{jp\omega t}=F_0e^{j\omega t}+s_1\sum_{p=1}^{\infty}\left(\delta_{2p}-\delta_{1p}\right)e^{jp\omega t}+s_3\left(\sum_{p=1}^{\infty}\left(\delta_{2p}-\delta_{1p}\right)e^{jp\omega t}\right)^3\\
&\sum_{p=1}^{\infty}p^2\omega^2 m\delta_{2p}e^{jp\omega t}=s_1\sum_{p=1}^{\infty}\left(\delta_{3p}+\delta_{1p}-2\delta_{2p}\right)e^{jp\omega t}+\\
&\qquad\qquad+s_3\left[\left(\sum_{p=1}^{\infty}\left(\delta_{3p}-\delta_{2p}\right)e^{jp\omega t}\right)^3-\left(\sum_{p=1}^{\infty}\left(\delta_{2p}-\delta_{1p}\right)e^{jp\omega t}\right)^3\right]\\
&\sum_{p=1}^{\infty}p^2\omega^2 m\delta_{3p}e^{jp\omega t}=s_1\sum_{p=1}^{\infty}\left(\delta_{2p}-\delta_{3p}\right)e^{jp\omega t}+s_3\left(\sum_{p=1}^{\infty}\left(\delta_{2p}-\delta_{3p}\right)e^{jp\omega t}\right)^3
\end{aligned} \tag{11}$$

As we can see, due to this substitution, the modelling system (equation 11) has gained the known form of the ensemble of equalities of harmonic components. So, as we said above, this system identically vanishes only with the equality of related coefficients for all harmonics.

To select the corresponding harmonics, we could make use of the orthogonality of the system of functions $e^{jp\omega t}$, $p = 1, 2, 3, \ldots$ (see, e.g., [20, p. 213]) and time independence of δ_{ip}, applying to equation 11 the operation similar to the ‘convolution’ usually used in selecting the Fourier harmonics. Since δ_{ip} are some analytical functions which can turn into infinity at the resonance frequencies, the substantiation of the applicability of convolution in this case will be incomplete. However, in systems of the type to which equation 11 applies, there is no need to apply the convolution to select the harmonic components: it is sufficient to set to zero the coefficients at the corresponding harmonics in each equality of this system. With this approach, the resonance pattern of the coefficients δ_{ip} will be inessential if the coefficients of related harmonics in each equality of equation 11 in the aggregate are identically zero.

Using this standard algebraic technique to select the harmonics, consider sequentially the systems of equations in 11 for each harmonic separately.

For the first harmonic (at $p = 1$), a strictly linear system of equations is formed:

$$\begin{aligned} -\omega^2 m\delta_{11} &= F_0 + s_1(\delta_{21} - \delta_{11}) \\ -\omega^2 m\delta_{21} &= s_1(\delta_{31} + \delta_{11} - \delta_{21}) \\ -\omega^2 m\delta_{31} &= s_1(\delta_{21} - \delta_{31}) \end{aligned} \tag{12}$$

We already know its solutions, as they are similar to equation 5:

$$\begin{aligned} \delta_{i1} &= -\frac{F_0 \cos(7-2i)\tau_1}{\omega\sqrt{s_1 m}\sin 6\tau_1}, \quad \beta_1 < 1 \\ \delta_{i1} &= (-1)^i \frac{F_0\left(\gamma_{1+}^{7-2i} + \gamma_{1-}^{7-2i}\right)}{\omega\sqrt{s_1 m}\left(\gamma_{1+}^6 - \gamma_{1-}^6\right)}, \quad \beta_1 > 1 \\ \delta_{i1} &= (-1)^i \frac{F_0(7-2i)}{12 s_1}, \quad \beta_1 = 1 \end{aligned} \tag{13}$$

where $i = 1, 2, 3$:

$$\beta_1 = \sqrt{\frac{\omega^2 m}{4 s_1}}; \quad \tau_1 = \arcsin\beta_1; \quad \gamma_{1+} = \sqrt{\beta_1^2 - 1} + \beta_1; \quad \gamma_{1-} = \sqrt{\beta_1^2 - 1} - \beta_1 \tag{14}$$

Note that in non-linear dynamics all three vibration regimes remain for the first harmonic, and the boundary frequency, ω_{01}, also corresponds to the linear vibration regime:

$$\omega_{01} = 2\sqrt{\frac{s_1}{m}} \tag{15}$$

The system of equations for the second harmonic is the following:

$$
\begin{aligned}
-4\omega^2 m\delta_{12} &= s_1(\delta_{22}-\delta_{12})\\
-4\omega^2 m\delta_{22} &= s_1(\delta_{32}+\delta_{12}-2\delta_{22})\\
-4\omega^2 m\delta_{32} &= s_1(\delta_{22}-\delta_{32})
\end{aligned}
\tag{16}
$$

This system is also linear and relates to free vibrations in an elastic line of stiffness s_1. We can see from equation 16 that where the applied force, $F(t)$, does not have a second harmonic, and/or the stiffness coefficient does not have a square term of the expansion, and/or the free vibrations are absent in the system by the statement of problem, the equality

$$
\delta_{i2} = 0, i = 1, 2, 3 \tag{17}
$$

is true. However, this result is not general. If any of the above conditions is violated (not only the square term in the expansion of the powers of the stiffness coefficient is absent, as is conventional), the second harmonic is present in the non-linear dynamic process, causing the hysteresis in the vibration pattern. It is desirable to take this feature into account in solving the problems of non-linear dynamics.

The system of equations for the third harmonic can be written as:

$$
\begin{aligned}
-9\omega^2 m\delta_{13} &= s_1(\delta_{23}-\delta_{13})+Q_{13}\\
-9\omega^2 m\delta_{23} &= s_1(\delta_{33}+\delta_{13}-2\delta_{23})+Q_{23}\\
-9\omega^2 m\delta_{33} &= s_1(\delta_{23}-\delta_{33})+Q_{33}
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
Q_{13} &= s_3(\delta_{21}-\delta_{11})^3\\
Q_{23} &= s_3\left[(\delta_{31}-\delta_{21})^3-(\delta_{21}-\delta_{11})^3\right]\\
Q_{33} &= s_3(\delta_{21}-\delta_{31})^3
\end{aligned}
\tag{19}
$$

As we already know all values of δ_{i1} involved in Q_{i3}, equation 18 is also reduced to the linear system describing the vibrations in an elastic line of stiffness s_1 (the same as for the first harmonic). The non-linear coefficient s_3 itself determines the amplitude of equivalent forces Q_{i3} affecting the corresponding elements of an elastic line. Except s_3, the momentary shifts δ_{i1} are involved in equation 19; the direct dependence of the vibration pattern of the third harmonic on the vibration amplitude of the first harmonic corroborates this. As the expressions in equation 19 show, this dependence is cubic.

To follow the analysis conveniently, slightly transform equation 19, using the values of the momentary shifts (equation 13) that we have found before. At $\beta_1 < 1$ we yield:

$$Q_{13} = s_3\left(\frac{F_0}{s_1 \sin 6\tau_1}\right)^3 \sin^3 4\tau_1$$

$$Q_{23} = s_3\left(\frac{F_0}{s_1 \sin 6\tau_1}\right)^3 \left(\sin^3 2\tau_1 - \sin^3 4\tau_1\right) \tag{20}$$

$$Q_{33} = -s_3\left(\frac{F_0}{s_1 \sin 6\tau_1}\right)^3 \sin^3 2\tau_1$$

at $\beta_1 > 1$

$$Q_{13} = -s_3\left(\frac{F_0}{s_1\left(\gamma_{1+}^6 - \gamma_{1-}^6\right)}\right)^3 \left(\gamma_{1+}^4 - \gamma_{1-}^4\right)^3$$

$$Q_{23} = s_3\left(\frac{F_0}{s_1\left(\gamma_{1+}^6 - \gamma_{1-}^6\right)}\right)^3 \left[\left(\gamma_{1+}^4 - \gamma_{1-}^4\right)^3 + \left(\gamma_{1+}^2 - \gamma_{1-}^2\right)^3\right] \tag{21}$$

$$Q_{33} = -s_3\left(\frac{F_0}{s_1\left(\gamma_{1+}^6 - \gamma_{1-}^6\right)}\right)^3 \left(\gamma_{1+}^2 - \gamma_{1-}^2\right)^3$$

and at $\beta_1 = 1$

$$Q_{13} = -s_3\left(\frac{2}{3}\right)^3\left(\frac{F_0}{s_1}\right)^3; \quad Q_{23} = s_3\,\frac{1}{3}\left(\frac{F_0}{s_1}\right)^3; \quad Q_{23} = -s_3\left(\frac{1}{3}\right)^3\left(\frac{F_0}{s_1}\right)^3 \tag{22}$$

We can see from equations 20–22 that the pattern of the equivalent forces Q_{i3} outside the resonance region depends not only on the amplitude of the first harmonic of momentary shifts, but also on the pattern of vibration regime in which the first harmonic is present. If the first harmonic is in the periodical vibration regime, all resonances arising in the first harmonic are reflected in the third harmonic. If the first harmonic is in the aperiodical vibration regime, the amplitude of equivalent forces of the third harmonic will rapidly decline as frequency increases; in this problem it is approximately proportional to β^{-18}.

On the basis of equations 20–22, we can determine the convergence conditions of the series (equation 7) outside the resonance region. For the third harmonic, in absence of the second, this is determined by the criterion of amplitude smallness of equivalent forces of this harmonic in relation to the amplitude of the applied force. In order to demonstrate it, it is sufficient to present s_1 and s_3 through the stiffness of an elastic line T_1 and T_3:

$$s_1 = \frac{T_1}{a}; \quad s_3 = \frac{T_3}{a^3}; \quad T_3 < T_1 \tag{23}$$

where a is the distance between the non-excited elements of an elastic line. Then, under

$$T_3 < T_1 \tag{24}$$

given the condition of non-destructive vibrations in an elastic line is

$$T_1 > F_0 \tag{25}$$

we obtain with equation 20 the following:

$$s_3\left(\frac{F_0}{s_1}\right)^3 = T_3\left(\frac{F_0}{T_1}\right)^3 = F_0\left(\frac{F_0}{T_1}\right)^2 \frac{T_3}{T_1} < F_0 \tag{26}$$

Thus we can see that, out of the resonance frequency and satisfying the inequality 24, the third harmonic corresponds to series decreasing in amplitude. However, if inequality 24 is not true, then even with the remaining $s_3 < s_1$, the convergence of equation 7 will depend on the amplitude of the applied force. At the condition

$$F_0 < \sqrt{\frac{T_1^3}{T_3}}, \quad \text{or} \quad F_0 < \sqrt{\frac{s_1^3}{s_3}} \tag{27}$$

following from equations 23 and 26, the third harmonic's amplitude will correspond to the convergent series. In the opposite case, outside the resonance band, we can observe the growing amplitude of the terms in equation 7, corresponding to the transition to an unstable process. Indeed, this criterion is very approximate and rather demonstrates the presence of a stability criterion for the vibration process, and, in generalisation, taking into account the terms of equation 7, it can be made more precise. Nonetheless, even from this initial evaluation, we can see that in non-linear systems the amplitude of the applied force influences the stability of dynamic processes.

In order to find the solutions to equation 18 for the third harmonic, taking into consideration the non-linearity of this system of equations, it is sufficient to present a more general form of solution as a linear superposition of three ancillary solutions for the subsystems of equations while retaining one of the equivalent forces, Q_{i3}, in each of them. With it the general solution takes the form:

$$\delta_{i3} = \sum_{r=1}^{3} \delta_{i3}^{r} \tag{28}$$

In turn, we can determine the solutions for each subsystem for the elastic line on the basis of the results presented in [14].

With it, for the first subsystem, containing the force Q_{13}, we yield:

$$\begin{aligned} \delta_{i3}^1 &= -\frac{Q_{13}\cos(7-2i)\tau_3}{3\omega\sqrt{s_1 m}\,\sin 6\tau_3}, \quad \beta_3 < 1 \\ \delta_{i3}^1 &= (-1)^i \frac{Q_{13}\left(\gamma_{3+}^{7-2i} + \gamma_{3-}^{7-2i}\right)}{3\omega\sqrt{s_1 m}\left(\gamma_{3+}^6 - \gamma_{3-}^6\right)}, \quad \beta_3 > 1 \\ \delta_{i3}^1 &= (-1)^i \frac{Q_{13}(7-2i)}{36\omega s_1}, \quad \beta_3 = 1 \end{aligned} \tag{29}$$

For the second subsystem, containing the force Q_{23}, the solution has some other form, because in this case the force affects the interior element of the elastic line. Should the line have more than three elements, according to [14] the solution would be the system for the right and left parts of the elastic line, correspondingly. But, since our model has only one interior element, the solution is simplified; it can be written as:

$$\delta_{i3}^2 = -\frac{Q_{13}\cos(2i-1)\tau_3 \cos 3\tau_3}{3\omega\sqrt{s_1 m}\cos\tau_3 \sin 6\tau_3}, \quad \beta_3 < 1$$

$$\delta_{i3}^2 = (-1)^i \frac{Q_{13}\left(\gamma_{3+}^{2i-1} + \gamma_{3-}^{2i-1}\right)\left(\gamma_{3+}^3 + \gamma_{3-}^3\right)}{3\omega\sqrt{s_1 m}\left(\gamma_{3+} + \gamma_{3-}\right)\left(\gamma_{3+}^6 - \gamma_{3-}^6\right)}, \quad \beta_3 > 1 \tag{30}$$

$$\delta_{i3}^2 = (-1)^i \frac{Q_{13}(2i-1)}{12\omega s_1}, \quad \beta_3 = 1$$

Finally, for the third subsystem, the solution has the following form:

$$\delta_{i3}^3 = -\frac{Q_{13}\cos(2i-1)\tau_3}{3\omega\sqrt{s_1 m}\cos 6\tau_3}, \quad \beta_3 < 1$$

$$\delta_{i3}^3 = (-1)^i \frac{Q_{13}\left(\gamma_{3+}^{2i-1} + \gamma_{3-}^{2i-1}\right)}{3\omega\sqrt{s_1 m}\left(\gamma_{3+}^6 - \gamma_{3-}^6\right)}, \quad \beta_3 > 1 \tag{31}$$

$$\delta_{i3}^3 = (-1)^i \frac{Q_{13}(2i-1)}{36\omega s_1}, \quad \beta_3 = 1$$

The obtained components (equations 29–31) of the general solution (equation 28) show that the amplitude–frequency characteristic of the vibration process for the third harmonic does not correspond to that of the first harmonic (different resonance frequencies are determined). This is conditioned not only by the differences of regularities. After all, the multiplier $\sin 6\tau_3$ determining the location of resonances has the same form as in equation 13 for the first harmonic, but for the third harmonic the parameter β_3, determining $\tau_3 = \arcsin\beta_3$, is equal to:

$$\beta_3 = 3\sqrt{\frac{\omega^2 m}{4 s_1}} = 3\beta_1 \tag{32}$$

This is three times greater than the related parameter (equation 14) of the first harmonic. Consequently, the boundary frequency for the third harmonic is one-third of ω_{01} and determined by the expression:

$$\omega_{03} = \frac{2}{3}\sqrt{\frac{s_1}{m}} = \frac{\omega_{01}}{3} \tag{33}$$

In its turn, this causes a threefold narrowing of the frequency band in the periodical regime for the third harmonic, which retains the general number of resonance peaks, now located on the lower third of the band of the first harmonic. At higher frequencies than the boundary ω_{03}, the vibrations of the third harmonic correspond to aperiodical damping along the line vibration regime and are localised in the regions of application of the equivalent forces. At the same time, the periodical vibration regime of the first harmonic remains and affects the vibration amplitude of the third harmonic through the value of equivalent forces Q_{i3} in accordance with equations 29–31. Due to this, in the present case, despite the non-resonant vibration pattern in the aperiodical vibration regime, at the overcritical range of the third harmonic there arise the resonances introduced from the first harmonic. These resonances are limited by the regions affected by the equivalent forces, because they arise on the background of vibrations of the third harmonic, effectively damping in space, which is typical for the aperiodical regime.

To reveal the general regularities of the next harmonics, we shall determine the fourth harmonic. On the basis of equation 11, this system of equations has the following form:

$$\begin{aligned} -16\omega^2 m\delta_{14} &= s_1(\delta_{24}-\delta_{14})+Q_{14} \\ -16\omega^2 m\delta_{24} &= s_1(\delta_{34}+\delta_{14}-2\delta_{24})+Q_{24} \\ -16\omega^2 m\delta_{34} &= s_1(\delta_{24}-\delta_{34})+Q_{34} \end{aligned} \tag{34}$$

where

$$\begin{aligned} Q_{14} &= s_3(\delta_{21}-\delta_{11})^2(\delta_{22}-\delta_{12}) \\ Q_{24} &= s_3\left[(\delta_{31}-\delta_{21})^2(\delta_{32}-\delta_{22})-(\delta_{21}-\delta_{11})^3(\delta_{22}-\delta_{12})\right] \\ Q_{34} &= s_3(\delta_{21}-\delta_{31})^2(\delta_{22}-\delta_{32}) \end{aligned} \tag{35}$$

The first point to notice is that the structure of equation 35 fully coincides with the structure of equation 18 for the third harmonic. However, the equivalent forces, Q_{i4}, have another appearance. Their amplitude is determined by the pattern of vibration of both the first and second harmonics. So, if the equality in equation 17 is true, the equality

$$\delta_{i4}=0; \quad i=1,2,3 \tag{36}$$

is also true. However, if equation 17 is not true, then, due to the general structure of equations 18 and 34, the solution of equation 34 is similar to that for equation 28, with the substitution of Q_{i3} into Q_{i4} and changing the coefficient at ω from 3 to 4. With it, the parameter β_4 changes: it becomes four times greater than β_1, and the boundary frequency ω_{04} becomes a quarter of ω_{01}, too. All features of the resonances described above for the third harmonic remain true for this fourth.

Generalising this investigation of four harmonics, we can state that for all the following harmonics the structure of the system of equations will remain, and we can represent it in the following form:

$$\begin{aligned} -(p\omega)^2 m\delta_{1p} &= s_1\left(\delta_{2p}-\delta_{1p}\right)+Q_{1p} \\ -(p\omega)^2 m\delta_{2p} &= s_1\left(\delta_{3p}+\delta_{1p}-2\delta_{2p}\right)+Q_{2p} \\ -(p\omega)^2 m\delta_{3p} &= s_1\left(\delta_{2p}-\delta_{3p}\right)+Q_{3p} \end{aligned} \tag{37}$$

The parameter β_p will be

$$\beta_p = p\sqrt{\frac{\omega^2 m}{4s_1}} = p\beta_1 \tag{38}$$

The boundary frequency of the *p*th harmonic will be *p* times less than that of the first harmonic:

$$\omega_{03} = \frac{2}{p}\sqrt{\frac{s_1}{m}} = \frac{\omega_{01}}{p} \tag{39}$$

The general solution for all harmonics higher than the first is similar to equations 29–31, with the corresponding substitution of equivalent forces and the multiplier at ω. Due to this, the resonance lines of each harmonic consist of the lines of natural resonances located before the boundary frequency of this harmonic, and of the introduced resonances from the lower harmonics, which are located between the natural boundary frequency of this harmonic and that of the first harmonic. The general structure of solutions for higher harmonics enables us to conclude that the condition of convergence of the series in equation 7 outside the resonance area will be similar to the condition obtained for the third harmonic (equation 27). In the resonance area, the amplitudes of harmonics follow the resonance curves of all lower harmonics and introduce their own resonance frequencies. So the convergence of the series in equation 7 on the whole for an elastic non-resistant line will be determined in fact only by the areas outside resonance. In the resonance areas, all harmonics will turn into infinity at the frequencies corresponding to the frequencies of the previous harmonics.

We can note that Landau and Lifshitz [22], studying the anharmonic vibrations, also came to the recurrent relation, when expanding the Lagrange function up to the third-power terms. For example, for normal vibrations in the second approximation they obtained:

$$\ddot{\Theta}_\alpha^{(2)} + \omega_\alpha^2 \Theta_\alpha^{(2)} = f_\alpha\left(\Theta^{(1)}, \dot{\Theta}^{(1)}, \ddot{\Theta}^{(1)}\right) \tag{40}$$

where $\Theta^{(1)}$ and $\Theta^{(2)}{}_a$ are the normal coordinates of the first and second approximations, respectively [22, p. 110]. However, in the absence of exact analytical solutions for the first approximation (the first harmonic), their further investigation was limited by seeking the combinational frequencies of higher harmonics. Because of this, with their approach, it is impossible to reveal the boundary frequency decrease with the growing number of harmonics in the presence of three vibration regimes for each harmonic, nor to reveal the introduced resonances, nor to reveal the analytical dependence of the vibration amplitudes of harmonics on the parameters of the system studied, nor to present the spectral expansion as a functional series. In contrast, in the recurrent relationship based on the exact analytical solutions, the equivalent forces are not reduced to the first and second derivatives of normal coordinates, but depend only on the amplitude shifts of lower harmonics, and the resonance vibrations of higher harmonics have a much more complex structure than the simplified form of combinational frequencies. It makes the spectral analysis of non-linear dynamic processes much more exact and complete. The main distinction is that, with the method presented, we do not have to solve the original problem for each harmonic: it is sufficient to have the general solution for the linear system of equations of the type shown as equation 37 to find recurrently the functional regularities for any harmonic. Furthermore, this is convenient in terms of numerical programming.

Prospects for further development of this method

As the obtained solution is common for all harmonics of a dynamic process, it allows (on the basis of the exact analytical solution of the linear problem) the spectral dispersion to be sequentially obtained of the momentary shift of each element of a non-linear elastic system in the analytical form. The revealed recurrent relationship is not limited to a specifically chosen simple modelled system. It would be sufficient to look at the broad complex of modelling differential equations presented in [9–16] to realise that they have a similar structure. It means that, however complex a model of an elastic line we investigate, if we approach the modelling in terms of the complete description of a *specific* model, which allows us to obtain the exact analytical solutions for the corresponding linear model, we will certainly obtain a sequential series of systems of linear equations for the harmonics of the particular process. The reason is that the stiffness coefficient expansion in powers will contain a linear part, which will form the linear system of equations for each harmonic of the spectral dispersion. Still, the power terms of spectral dispersion in any case will depend only on the momentary shifts of the lower harmonics, so they can be presented in the form of equivalent forces connecting the dynamic processes in the studied harmonic with the lower harmonics.

The range of validity of the obtained recurrent relationships is not limited by the stiffness coefficient non-linearity. The method works in the presence of non-linear resistance, in its expansion into the power series in terms of the shift velocity of the line elements, and in the case of the complex spectral composition of an external force. Similarly, the line can be heterogeneous and can contain, for example, the non-linear constraints in only some part of the elastic connections. In this case, the equivalent forces remain only in those equations of systems of all harmonics which describe the non-linear constraints, and the general structure of the solution remains.

The main advantage of the obtained solutions is their exactness. As we showed in the course of solution, in finding the harmonics of a dynamic process we did not use the asymptotic approximations, nor conditions of smallness, except the standard approximation of the stiffness coefficient by the power series. For each harmonic the solution was found by the exact analytical method, since in each case, without lessening generalisation, the problem was reduced to the linear system of equations with exact analytical solutions. As a result, we have obtained the functional series, which is not necessarily descending, and, outside the resonance, the inequality in equation 27 determines its convergence. Furthermore, we have to take into account that in an ideal line the resonance amplitude is infinite, and the general decrease in amplitude with the higher-order harmonics does not affect its value. The additional feature is that, in the general solution of ideal lines, the density of resonance frequencies increases as frequency decreases, and we can prove that at $\omega = 0$ this density turns to infinity. Such a problem is easy to overcome when taking into account the resistance present in every real elastic system. In this case, due to the finite amplitude of resonance [11], the increase in resonance density is compensated by an abrupt decrease in their amplitude, due to which the infinite spectral series can be limited by the given accuracy of the solution.

Finally, we would like to mention one more property of the solutions obtained. As we saw above, in order to find sequentially the spectral harmonics, it is sufficient to have the solution for one linear problem corresponding to that non-linear system. Thus, the problem of sequentially finding the higher harmonics of the vibration process is not complicated by the higher number of a harmonic, since, in order to find any high-order harmonic, it is sufficient to have a general algorithm of the solution, substituting into it sequentially the corresponding values of the equivalent forces and the number of the harmonic. This is the advantage provided by the analytical methods, as described by Cherepennikov [8], to whom we referred in the introduction.

Conclusions

We have revealed that, generally, the solution to a non-linear elastic constraint problem can be presented as a spectral functional series whose each harmonic is the solution of a linear system of equations for an elastic line with the stiffness coefficient equal to the linear term of expansion of this parameter into a power series in the amplitude of constraint deformation. The degree of non-linearity of an elastic constraint and the vibration amplitude of lower harmonics affects the vibration pattern of each harmonic. With the higher number of the harmonic, its boundary frequency diminishes proportionally to the harmonic number. The resonance frequency spectrum of each harmonic contains the spectrum of natural frequencies located lower than the boundary frequency, and the spectrum of the introduced resonances of lower harmonics located between the natural boundary frequency and the boundary frequency of the first harmonic.

We have ascertained that, outside the resonance band, the harmonic's amplitude decreases as its number increases, but in the case of ideal systems it does not affect the resonance amplitude. Noting that in the general solution the density of resonance frequencies increases up to infinity with the frequency of the process tending to zero, it will be more efficient to take the line resistance into account at once, in order to describe the dynamic processes more accurately. The reason is that, with the finite resonance amplitudes being typical for a resistant line, the growth of density of resonance frequencies is compensated by the fast decrease of their amplitude.

We have shown that this technique of the recurrent finding of the harmonics of a dynamic process can be extended to models with a non-linear resistance and to the case where the applied force has a complex spectral composition.

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