
Coping with curvilinear coordinates in fluid mechanics

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Abstract The method of local Cartesian coordinates introduced in the first part of a series of three articles on curvilinear coordinates was applied in the second article to some solid mechanics problems. In this final, third article of the series, the method is applied to fluid mechanics and for intrinsic coordinates. The use of the principle of virtual power is advocated to generate the appropriate equations of motion.

Keywords curvilinear coordinates, fluid mechanics, rate of strain, equations of motion

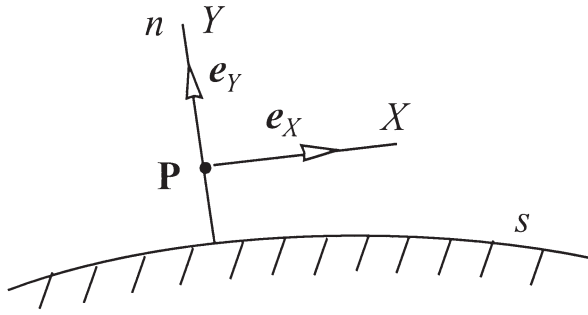
Introduction

The method of local Cartesian coordinates introduced in the first [1] of a series of three articles on curvilinear coordinates can be applied naturally to fluid mechanics in a similar way to that shown for problems of solid mechanics in the second article [2]. Therefore we consider only one application here, in this final, third article of the series. In the second article the coordinate systems employed were Lagrangian ones, as is conventional in solid mechanics. Now they are Eulerian [3, p. 138], as is usual in fluid mechanics. The equivalent of the principle of virtual work in fluid mechanics is the principle of virtual power. It is surprising that the former principle is presented in nearly every solid mechanics textbook but the latter is hardly ever mentioned in the fluid mechanics literature. This situation is likely to change gradually as the finite element method – using as its starting point so-called weak forms or formulations consisting of integral statements – gains more ground in fluid mechanics.

In this article we do not generate any ‘general’ formulas, as this would lead to too many repetitions of what has been done in the second [2]. Here we just apply the method of local Cartesian coordinates directly to the so-called intrinsic coordinate system (Fig. 1) [e.g. 4, p. 197; 5, p. 67]. The notations are almost the same as in Fig. 1 in Paavola and Salonen [2]. Coordinate s is now the arc length measured along the body surface and coordinate n is directed 90° anticlockwise from the positive direction of s . The curvature $1/R$ of the surface is positive when the wall is convex outwards.

We have again (see (12) and (14) in Paavola and Salonen [2])

$$\frac{d\mathbf{e}_s}{ds} = -\frac{1}{R}\mathbf{e}_n, \quad \frac{d\mathbf{e}_n}{ds} = \frac{1}{R}\mathbf{e}_s \quad (1)$$

Fig. 1 *Body intrinsic coordinates.*

and the scale factors

$$h_s = 1 + \frac{n}{R}, \quad h_n = 1 \quad (2)$$

At the local origin

$$\mathbf{e}_X = \mathbf{e}_s, \quad \mathbf{e}_Y = \mathbf{e}_n \quad (3)$$

and

$$\frac{\partial}{\partial X} = \left(1 + \frac{n}{R}\right)^{-1} \frac{\partial}{\partial s}, \quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial n} \quad (4)$$

Kinematics of flow, particle acceleration

Starting directly with fluids in motion, we need the expression for the acceleration in the Eulerian representation which is [5, p. 48]:

$$\mathbf{a} \equiv \frac{D\mathbf{u}}{Dt} \equiv \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \quad (5)$$

where \mathbf{u} is now the velocity and t the time. In two dimensions the velocity field has the representations:

$$\begin{aligned} \mathbf{u}(\mathbf{r}, t) &= u(x, y, t) \mathbf{i} + v(x, y, t) \mathbf{j} \\ &= u_X(X, Y, t) \mathbf{e}_X + u_Y(X, Y, t) \mathbf{e}_Y \\ &= u_\alpha(\alpha, \beta, t) \mathbf{e}_\alpha(\alpha, \beta) + u_\beta(\alpha, \beta, t) \mathbf{e}_\beta(\alpha, \beta) \end{aligned} \quad (6)$$

similarly to equations (1) in Paavola and Salonen [2].

For the intrinsic coordinates:

$$\mathbf{u}(s, n, t) = u_s(s, n, t) \mathbf{e}_s(s) + u_n(s, n, t) \mathbf{e}_n(s) \quad (7)$$

In Cartesian coordinates the acceleration components are:

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \\ a_y &= \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \end{aligned} \quad (8)$$

In the local system, similarly,

$$\begin{aligned} a_x &= \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial X} + u_y \frac{\partial u_x}{\partial Y} = \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{e}_x + (\mathbf{u} \cdot \mathbf{e}_x) \left(\frac{\partial \mathbf{u}}{\partial X} \cdot \mathbf{e}_x \right) + (\mathbf{u} \cdot \mathbf{e}_y) \left(\frac{\partial \mathbf{u}}{\partial Y} \cdot \mathbf{e}_x \right) \\ a_y &= \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial X} + u_y \frac{\partial u_y}{\partial Y} = \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{e}_y + (\mathbf{u} \cdot \mathbf{e}_x) \left(\frac{\partial \mathbf{u}}{\partial X} \cdot \mathbf{e}_y \right) + (\mathbf{u} \cdot \mathbf{e}_y) \left(\frac{\partial \mathbf{u}}{\partial Y} \cdot \mathbf{e}_y \right) \end{aligned} \quad (9)$$

Application of formulas (3) and (4) gives, first:

$$\begin{aligned} a_s &= \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{e}_s + (\mathbf{u} \cdot \mathbf{e}_s) \left(1 + \frac{n}{R} \right)^{-1} \left(\frac{\partial \mathbf{u}}{\partial s} \cdot \mathbf{e}_s \right) + (\mathbf{u} \cdot \mathbf{e}_n) \left(\frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{e}_s \right) \\ a_n &= \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{e}_n + (\mathbf{u} \cdot \mathbf{e}_s) \left(1 + \frac{n}{R} \right)^{-1} \left(\frac{\partial \mathbf{u}}{\partial s} \cdot \mathbf{e}_n \right) + (\mathbf{u} \cdot \mathbf{e}_n) \left(\frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{e}_n \right) \end{aligned} \quad (10)$$

The derivatives needed are:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= \frac{\partial u_s}{\partial t} \mathbf{e}_s + \frac{\partial u_n}{\partial t} \mathbf{e}_n \\ \frac{\partial \mathbf{u}}{\partial s} &= \frac{\partial u_s}{\partial s} \mathbf{e}_s - \frac{u_s}{R} \mathbf{e}_n + \frac{\partial u_n}{\partial s} \mathbf{e}_n + \frac{u_n}{R} \mathbf{e}_s = \left(\frac{\partial u_s}{\partial s} + \frac{u_n}{R} \right) \mathbf{e}_s + \left(\frac{\partial u_n}{\partial s} - \frac{u_s}{R} \right) \mathbf{e}_n \\ \frac{\partial \mathbf{u}}{\partial n} &= \frac{\partial u_s}{\partial n} \mathbf{e}_s + \frac{\partial u_n}{\partial n} \mathbf{e}_n \end{aligned} \quad (11)$$

Substitution in (10) gives:

$$\begin{aligned} a_s &= \frac{\partial u_s}{\partial t} + \left(1 + \frac{n}{R} \right)^{-1} u_s \left(\frac{\partial u_s}{\partial s} + \frac{u_n}{R} \right) + u_n \frac{\partial u_s}{\partial n} \\ a_n &= \frac{\partial u_n}{\partial t} + \left(1 + \frac{n}{R} \right)^{-1} u_s \left(\frac{\partial u_n}{\partial s} - \frac{u_s}{R} \right) + u_n \frac{\partial u_n}{\partial n} \end{aligned} \quad (12)$$

Rate of strain

The rate of strain components in Cartesian coordinates are [5, p. 53]:

$$\begin{aligned} d_x &= \frac{\partial u}{\partial x} \\ d_y &= \frac{\partial v}{\partial y} \\ g_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned} \quad (13)$$

These are the analogues of (2) in Paavola and Salonen [2]. In the local system:

$$\begin{aligned}d_x &= \frac{\partial u_x}{\partial X} = \frac{\partial \mathbf{u}}{\partial X} \cdot \mathbf{e}_x \\d_y &= \frac{\partial u_y}{\partial Y} = \frac{\partial \mathbf{u}}{\partial Y} \cdot \mathbf{e}_y \\g_{xy} &= \frac{\partial u_x}{\partial Y} + \frac{\partial u_y}{\partial X} = \frac{\partial \mathbf{u}}{\partial Y} \cdot \mathbf{e}_x + \frac{\partial \mathbf{u}}{\partial X} \cdot \mathbf{e}_y\end{aligned}\quad (14)$$

In the intrinsic coordinate system:

$$\begin{aligned}d_s &= \left(1 + \frac{n}{R}\right)^{-1} \frac{\partial \mathbf{u}}{\partial s} \cdot \mathbf{e}_s \\d_n &= \frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{e}_n \\g_{sn} &= \frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{e}_s + \left(1 + \frac{n}{R}\right)^{-1} \frac{\partial \mathbf{u}}{\partial s} \cdot \mathbf{e}_n\end{aligned}\quad (15)$$

Substitution of the derivatives in (11) gives:

$$\begin{aligned}d_s &= \left(1 + \frac{n}{R}\right)^{-1} \left(\frac{\partial u_s}{\partial s} + \frac{u_n}{R}\right) \\d_n &= \frac{\partial u_n}{\partial n} \\g_{sn} &= \frac{\partial u_s}{\partial n} + \left(1 + \frac{n}{R}\right)^{-1} \left(\frac{\partial u_n}{\partial s} - \frac{u_s}{R}\right)\end{aligned}\quad (16)$$

Vorticity of flow

The vorticity for two-dimensional flow is defined in Cartesian coordinates as [5, p. 73]:

$$\omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (17)$$

Thus, in the local system:

$$\omega = \frac{1}{2} \left(\frac{\partial u_y}{\partial X} - \frac{\partial u_x}{\partial Y} \right) = \frac{1}{2} \left(\frac{\partial \mathbf{u}}{\partial X} \cdot \mathbf{e}_y - \frac{\partial \mathbf{u}}{\partial Y} \cdot \mathbf{e}_x \right) \quad (18)$$

and in the intrinsic system

$$\omega = \frac{1}{2} \left[\left(1 + \frac{n}{R}\right)^{-1} \frac{\partial \mathbf{u}}{\partial s} \cdot \mathbf{e}_n - \frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{e}_s \right] = \frac{1}{2} \left[\left(1 + \frac{n}{R}\right)^{-1} \left(\frac{\partial u_n}{\partial s} - \frac{u_s}{R} \right) - \frac{\partial u_s}{\partial n} \right] \quad (19)$$

Equations of motion from virtual power

The principle of virtual power has been presented and discussed by Connor and Brebbia [6, p. 164]. As mentioned in [6]: ‘Its role corresponds to that of the principle of virtual displacements in solid mechanics’.

The principle of virtual power can be expressed as:

$$\delta P^i + \delta P^e = 0 \quad (20)$$

Here, for a two-dimensional continuum, the virtual power of internal forces is:

$$\delta P^i = - \int_A (\sigma_x \delta d_x + \sigma_y \delta d_y + \tau_{xy} \delta g_{xy}) dA \quad (21)$$

The virtual power of external forces is:

$$\delta P^e = \int_A [\rho(b_x - a_x) \delta u + \rho(b_y - a_y) \delta v] dA + \int_{s_r} (t_x \delta u + t_y \delta v) ds \quad (22)$$

The virtual rate of strain components are connected to the virtual velocities by (again, the commutative law between variation and differentiation is used):

$$\begin{aligned} \delta d_x &= \frac{\partial \delta u}{\partial x} \\ \delta d_y &= \frac{\partial \delta v}{\partial y} \\ \delta g_{xy} &= \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \end{aligned} \quad (23)$$

The formulation is analogous to the virtual work formulation, the main differences being that we have replaced the body force intensity per unit volume, \mathbf{f} , with $\rho \mathbf{b}$, where \mathbf{b} is the intensity per unit mass, which is more usual in fluid mechanics, and the inertia forces are taken into account.

In the intrinsic coordinate system with $dA = h_s ds h_n dn = (1 + n/R) ds dn$:

$$\begin{aligned} \delta P^i &= - \int_A (\sigma_s \delta d_s + \sigma_n \delta d_n + \sigma_{sn} \delta g_{sn}) dA \\ &= - \int_{s,n} \left\{ \sigma_s \left(\frac{\partial \delta u_s}{\partial s} + \frac{\delta u_n}{R} \right) + \sigma_n \left(1 + \frac{n}{R} \right) \frac{\partial \delta u_n}{\partial n} \right. \\ &\quad \left. + \sigma_{sn} \left[\left(1 + \frac{n}{R} \right) \frac{\partial \delta u_s}{\partial n} + \left(\frac{\partial \delta u_n}{\partial s} - \frac{\delta u_s}{R} \right) \right] \right\} ds dn \end{aligned} \quad (24)$$

Integration by parts gives:

$$\begin{aligned} \delta P^i &= \int_{s,n} \left\{ \left[\frac{\partial \sigma_s}{\partial s} + \frac{\partial}{\partial n} \left[\left(1 + \frac{n}{R} \right) \sigma_{sn} \right] + \frac{\sigma_{sn}}{R} \right] \delta u_s + \right. \\ &\quad \left. \left[- \frac{\sigma_s}{R} + \frac{\partial}{\partial n} \left[\left(1 + \frac{n}{R} \right) \sigma_n \right] + \frac{\partial \sigma_{sn}}{\partial s} \right] \delta u_n \right\} ds dn + bt \end{aligned} \quad (25)$$

where bt refers to boundary terms, which we are not interested in here, as we want to derive only the field equations. The virtual power of external forces is:

$$\begin{aligned} \delta P^e &= \int_A [\rho(b_s - a_s)\delta u_s + \rho(b_n - a_n)\delta u_n] dA + bt \\ &= \int_{s,n} [\rho(b_s - a_s)\delta u_s + \rho(b_n - a_n)\delta u_n] \left(1 + \frac{n}{R}\right) dsdn + bt \end{aligned} \quad (26)$$

Application of the principle of virtual power (20) gives the equations of motion:

$$\begin{aligned} \left(1 + \frac{n}{R}\right)\rho(b_s - a_s) + \frac{\partial \sigma_s}{\partial s} + \frac{\partial}{\partial n} \left[\left(1 + \frac{n}{R}\right)\sigma_{sn} \right] + \frac{\sigma_{sn}}{R} &= 0 \\ \left(1 + \frac{n}{R}\right)\rho(b_n - a_n) - \frac{\sigma_s}{R} + \frac{\partial}{\partial n} \left[\left(1 + \frac{n}{R}\right)\sigma_n \right] + \frac{\partial \sigma_{sn}}{\partial s} &= 0 \end{aligned} \quad (27)$$

Some manipulation gives:

$$\begin{aligned} \rho a_s &= \left(1 + \frac{n}{R}\right)^{-1} \frac{\partial \sigma_s}{\partial s} + \frac{\partial \sigma_{sn}}{\partial n} + 2 \left(1 + \frac{n}{R}\right)^{-1} \frac{\sigma_{sn}}{R} + \rho b_s \\ \rho a_n &= - \left(1 + \frac{n}{R}\right)^{-1} \frac{\sigma_s}{R} + \frac{\partial \sigma_n}{\partial n} + \left(1 + \frac{n}{R}\right)^{-1} \frac{\sigma_n}{R} + \frac{\partial \sigma_{sn}}{\partial s} + \rho b_n \end{aligned} \quad (28)$$

Further forms are obtained by performing the conventional stress tensor decomposition [5, p. 58]:

$$\begin{aligned} \sigma_s &= -p + \sigma'_s \\ \sigma_n &= -p + \sigma'_n \end{aligned} \quad (29)$$

where p is the pressure and σ'_n and σ'_s are deviatoric stresses. By introducing the expressions (12):

$$\begin{aligned} \rho \left[\frac{\partial u_s}{\partial t} + \left(1 + \frac{n}{R}\right)^{-1} u_s \left(\frac{\partial u_s}{\partial s} + \frac{u_n}{R} \right) + u_n \frac{\partial u_s}{\partial n} \right] &= \\ - \left(1 + \frac{n}{R}\right)^{-1} \frac{\partial p}{\partial s} + \left(1 + \frac{n}{R}\right)^{-1} \frac{\partial \sigma'_s}{\partial n} + \frac{\partial \sigma_{sn}}{\partial n} + 2 \left(1 + \frac{n}{R}\right)^{-1} \frac{\sigma_{sn}}{R} + \rho b_s & \quad (30) \\ \rho \left[\frac{\partial u_n}{\partial t} + \left(1 + \frac{n}{R}\right)^{-1} u_s \left(\frac{\partial u_n}{\partial s} - \frac{u_s}{R} \right) + u_n \frac{\partial u_n}{\partial n} \right] &= \\ - \frac{\partial p}{\partial n} - \left(1 + \frac{n}{R}\right)^{-1} \frac{\sigma'_n - \sigma'_s}{R} + \frac{\partial \sigma_{sn}}{\partial s} + \rho b_n & \end{aligned}$$

Concluding remarks

No constitutive relations such as Stokes' friction law (or Hooke's law in solid mechanics) have been introduced here, as they are not needed in kinematics or in

the principle of virtual power (or in the principle of virtual work). Of course, to close the equations (to render them solvable), constitutive relations are needed. By introducing this theme only after general kinematics and equilibrium equations or equations of motion have been derived should make the hierarchy of the theory as transparent as possible for the student. We again want to emphasize the central roles of the principle of virtual work and virtual power in the teaching of mechanics. As the increasingly important finite element method makes direct use of these principles, the student must in any case assimilate the principles to some extent. It is then quite logical to make additional use of them in analytical calculations.

In numerical applications, the method of local Cartesian coordinates can also be used in cases where the coordinate system is not orthogonal, as is the case, for instance, in the finite element method when, say, isoparametric elements are used [7].

The method of local Cartesian coordinates can naturally be applied in areas of physics and engineering other than those considered here for example in electricity and magnetism.

References

- [1] J. Paavola and E.-M. Salonen, 'Coping with curvilinear coordinates', *International Journal of Mechanical Engineering Education*, **26** (1998), 309–317.
- [2] J. Paavola and E.-M. Salonen, 'Coping with curvilinear coordinates in solid mechanics', *International Journal of Mechanical Engineering Education*, **32** (2004), 1–10.
- [3] L.E. Malvern, *Introduction to the Mechanics of a Continuous Medium* (Prentice-Hall, Englewood Cliffs, NJ, 1969).
- [4] D.A. Anderson, J.C. Tannehill and R.H. Pletcher, *Computational Fluid Mechanics and Heat Transfer* (McGraw-Hill, New York, 1984).
- [5] H. Schlichting, *Boundary-Layer Theory* (McGraw-Hill, New York, 1979).
- [6] J.J. Connor and C.A. Brebbia, *Finite Element Techniques for Fluid Flow* (Newnes-Butterworths, London, 1976).
- [7] J. Paavola, 'A study of curved thin-walled girders', *Acta Polytechnica Scandinavica*, **94** (1990), 91.