The *P–V* linear expansion of an ideal gas

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Abstract The heat transfer to and from a perfect gas, through a *P–V* linear expansion, is analysed as a pedagogic exercise. The *T–S* analysis proves to be useful to calculate the thermodynamic efficiency of certain cycles and helps students to a better understanding of the second law and other fundamental topics.

Keywords ideal gas; *P–V* linear expansion; heat transfer; efficiency

Introduction

The negatively sloping expansion of an ideal gas requires some thought, as indicated by Dickerson and Mottman [1], so that typical problems can be solved correctly when teaching introductory thermodynamics. Working out the heat flow to and from the expanding gas can contribute to a better understanding of the second law by students; it is a didactic exercise that is also very useful in relation to other topics of fundamental thermodynamics as they are usually presented in classical textbooks [2–4].

Since the *P–V* linear expansion of the gas has already been comprehensively analysed in the *P–V* diagram in previous literature [1, 5–8], in this paper the expansion of the gas is tackled mainly following the *T–S* representation, in which the heat flow to and from the gas can be straightforwardly read.

The *P–V* diagram

If the straight line representing the linear expansion of an ideal gas runs from point $(0, P_0)$ to $(V_0, 0)$ its equation is:

$$P = P_0 \frac{V_0 - V}{V_0} \tag{1}$$

Now, only two results will be recalled from the *P–V* diagram with respect to linear expansions of an ideal gas. First, the boundary between the gas absorbing heat and delivering it happens at:

$$V = \frac{\gamma}{\gamma + 1} V_0 \tag{2}$$

which is the same for all the expansions converging on V_0 at $P = 0$, as represented in Fig. 1. Second, the variation of the temperature, while the gas increases its volume, is given by:

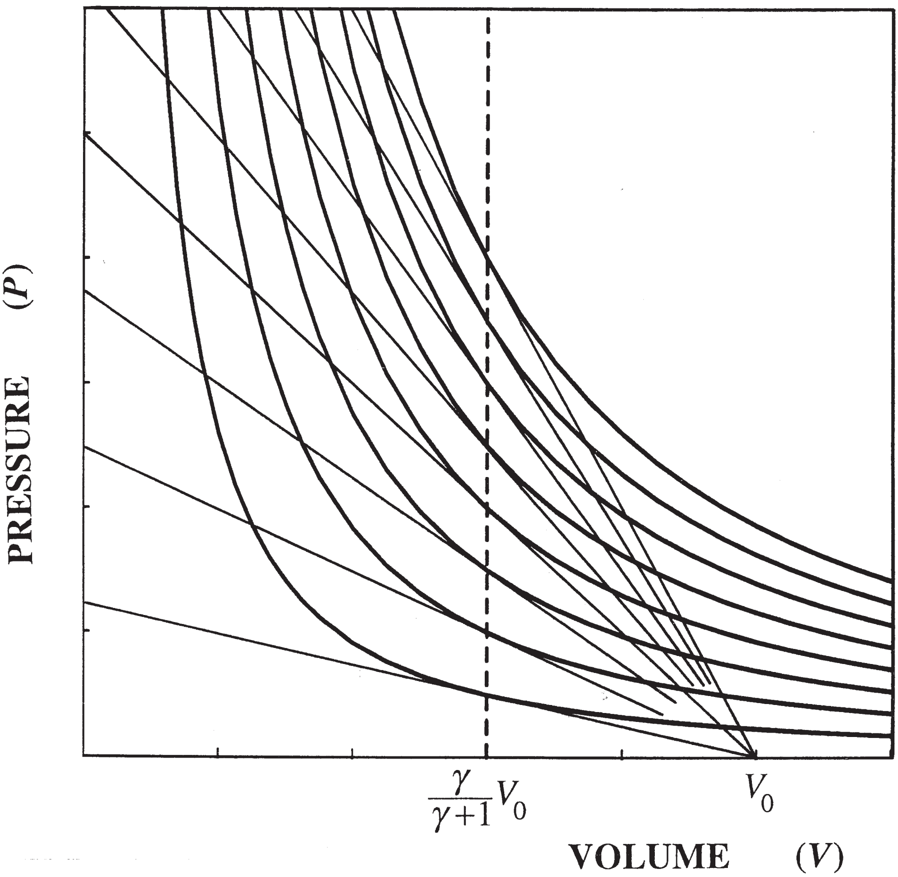

Fig. 1 *Linear expansions of an ideal gas converging on* V_0 *at* $P = 0$. *In all cases the heat energy stops entering the gas and starts leaving it at* $V = \gamma V_0/(\gamma + 1)$.

$$\frac{dT}{dV} = \frac{P_0}{NR}\frac{V_0 - 2V}{V_0} \tag{3}$$

P_0 being in each case the ordinate at $V = 0$.

The *T–S* diagram

Now, from a pedagogical point of view, it may be convenient, following the work by Valentine [9], to go further into the expansion of the ideal gas in the *T–S* diagram, in which the heat flow will be directly read. With this objective in mind, we write:

$$\delta Q = Nc_v dT + PdV \tag{4}$$

$$dS = \frac{\delta Q}{T} = Nc_v \frac{dT}{T} + \frac{p}{T} dV \tag{5}$$

and

$$\frac{dS}{dT} = \frac{1}{T}\left(Nc_v + P\frac{dV}{dT} \right) \tag{6}$$

Now, by substituting equations 1 and 3 into equation 6 and considering that the expanding gas accomplishes the equation of state:

$$PV = NRT \tag{7}$$

it is easy to get to the following result:

$$\frac{dT}{dS} = \frac{2}{N^2(\gamma - 1)c_V^2} \frac{P_0 V(V - V_0/2)(V - V_0)}{V_0 \gamma V_0 - (\gamma + 1)V} \tag{8}$$

Then, the integration of equation 8 is not necessary, but the information contained in Table 1, which can be drawn from it, will be sufficient to represent the expansion of the perfect gas in the *T–S* diagram.

Actually, following Table 1, the curve representing the expansion of the ideal gas described by equation 1 can be drawn in the *T–S* diagram, as shown in Fig. 2. In Fig. 2 it has been considered that the entropy vanishes at zero temperature, accord-

TABLE 1 Slope of the *P–V* linear expansion of an ideal gas on the *T–S* diagram, as deduced from equations 7 and 8

Volume, V		Slope, $\frac{dT}{dS}$
$V = 0$		$\frac{dT}{dS} = 0$
$0 < V < \frac{V_0}{2}$		$\frac{dT}{dS} > 0$
$V = \frac{V_0}{2}$	$T = \frac{1}{4}\frac{P_0 V_0}{NR}$	$\frac{dT}{dS} = 0$
$\frac{V_0}{2} < V < \frac{\gamma}{\gamma+1}V_0$		$\frac{dT}{dS} < 0$
$V = \frac{\gamma}{\gamma+1}V_0$	$T = \frac{\gamma}{(\gamma+1)^2}\frac{P_0 V_0}{NR}$	$\frac{dT}{dS} = \infty$
$\frac{\gamma}{\gamma+1}V_0 < V < V_0$		$\frac{dT}{dS} > 0$
$V = V_0$		$\frac{dT}{dS} = 0$

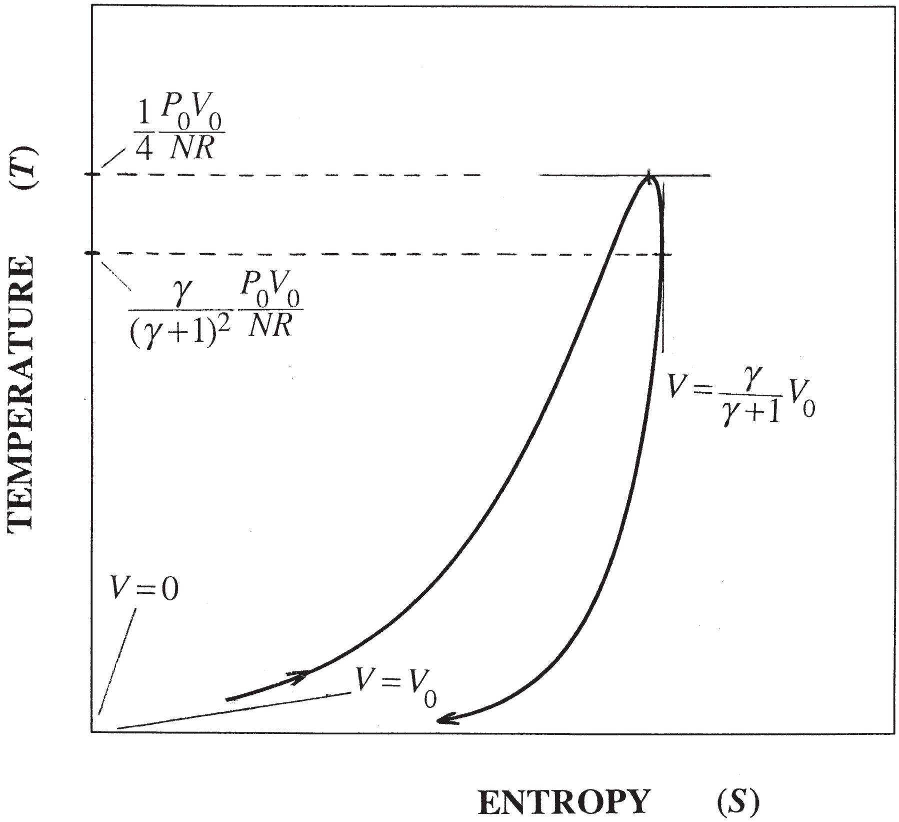

Fig. 2 P–V *linear expansion of the ideal gas on the* T–S *diagram, drawn from Table 1.*

ing to the Nernst postulate. Otherwise, the curve would remain identical in shape but be displaced on the *T–S* plane.

In Fig. 2 it is evident that the gas increases the entropy, which means it takes a heat flux from some reversible heat source, while

$$V < \frac{\gamma}{\gamma+1}V_0 \tag{9}$$

Afterwards, the entropy does not increase but diminishes and the gas starts delivering a heat flux.

Application

Let us consider now the motor cycle of Fig. 3, completed by a perfect gas, and try to work out the efficiency as:

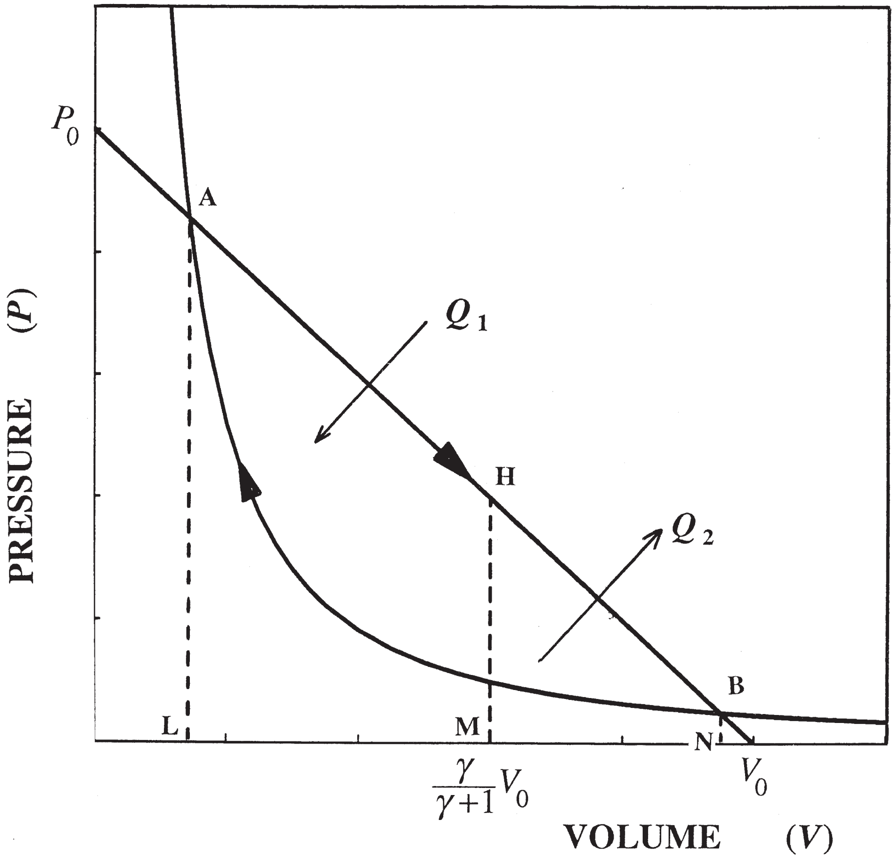

Fig. 3 *Reversible cycle completed by a perfect gas, following a linear expansion and an adiabatic compression.*

$$\eta = \frac{W}{Q_1} = 1 - \frac{Q_2}{Q_1} \tag{10}$$

where W is the work delivered by the gas system in each cycle, Q_1 is the heat flux the ideal gas receives throughout the cycle and Q_2 is the heat it delivers.

It is necessary to locate the point, H, at which the gas stops receiving heat and starts delivering it, before equation 10 can be applied. Then it is possible to calculate the amount of heat energy, Q_1, the gas takes from A to H and the fraction, Q_2, it delivers from H to B, providing work $W = Q_1 - Q_2$ per cycle.

Afterwards, the calculation of Q_1 and Q_2 can be carried out from the first law of thermodynamics, as follows:

$$Q_1 = Nc_v(T_H - T_A) + W_{AH} \tag{11}$$

and, similarly:

$$Q_2 = Nc_v(T_H - T_B) - W_{HB} \tag{12}$$

W_{AH} and W_{HB} being the amounts of work delivered by the gas expanding from A to H and from H to B, respectively, given by the areas *LAHM* and *MHBN*.

It is noted that, if the expansion *AB* were dealt with as a whole, equation 11 would yield as incorrect value of the heat flux entering the gas:

$$Q_1' = Nc_v(T_B - T_A) + W_{AB} = Q_1 - Q_2 \tag{13}$$

whereas the heat leaving the gas would appear to be null. This would lead to the erroneous result of the efficiency of the cycle being worth unity, which is against the second law of thermodynamics.

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