Coping with curvilinear coordinates in solid mechanics

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Abstract The method of local Cartesian coordinates introduced in the first of a series of three articles on curvilinear coordinates is applied in this second article to some problems of solid mechanics. Additionally, the principle of virtual work is combined with the geometry and the kinematics produced by the method of local Cartesian coordinates to generate appropriate equilibrium equations.

Keywords curvilinear coordinates, solid mechanics, strains, equilibrium equations

Introduction

A systematic method to produce expressions needed in physics and valid in curvilinear orthogonal coordinates was introduced in the first part [1] of a series of three articles on curvilinear coordinates. In brief, the idea explained in that first article is to employ a local Cartesian coordinate system or frame as a temporary tool to be used to advantage in curved geometries. First, as the frame is the familiar Cartesian one, the relevant expressions in a problem can be found in a simple way. The orientation of the frame at a given point can be selected at will, so that maximum benefit from the geometry of the problem is gained. Then, using the systematic steps described [1], the associated formulas in the curvilinear orthogonal coordinates emerge. The method is designed to faciliate the understanding of manipulations in curvilinear coordinates without the need of tensor analysis. It is also intended to replace in some measure the use of complicated differential geometry figures often used in textbooks. Though in simple cases these figures can be very useful, in complicated curved situations they may leave the student rather unsure about the correctness of the results. Further, in nonlinear cases the use of differential geometry figures is, in our opinion, almost hopeless.

In pure mathematics applications, one can always question the idea of deviating from the conventional derivations – for instance Arfken [2] – of the expressions for gradient, divergence, etc. In this second part of the series, the method of local Cartesian coordinates is applied to some problems of solid mechanics and specifically to the curved Timoshenko beam. Here, at least, we feel that the method has considerable value in teaching the subject.

Strains, general

The displacement field $\boldsymbol{u}$ has three alternative representations (cf. equation 18 in Paavola and Salonen [1]):

$$\begin{aligned}\boldsymbol{u} &= u\boldsymbol{i} + v\boldsymbol{j} \\ &= u_X\boldsymbol{e}_X + u_Y\boldsymbol{e}_Y \\ &= u_\alpha\boldsymbol{e}_\alpha + u_\beta\boldsymbol{e}_\beta\end{aligned} \tag{1}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{e}_X$, $\boldsymbol{e}_Y$ and $\boldsymbol{e}_\alpha$, $\boldsymbol{e}_\beta$ are unit vectors of a global Cartesian, a local Cartesian and a curvilinear coordinate systems, respectively. The well known expressions for small strain components in Cartesian coordinates are given, for example, by Timoshenko Goodier [3, p. 6]:

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\end{aligned} \tag{2}$$

If tensor notation is used, the shearing strain expression contains additionally the factor ½. In the local system we have:

$$\begin{aligned}\varepsilon_X &= \frac{\partial u_X}{\partial X} = \frac{\partial \boldsymbol{u}}{\partial X}\cdot\boldsymbol{e}_X \\ \varepsilon_Y &= \frac{\partial u_Y}{\partial Y} = \frac{\partial \boldsymbol{u}}{\partial Y}\cdot\boldsymbol{e}_Y \\ \gamma_{XY} &= \frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X} = \frac{\partial \boldsymbol{u}}{\partial Y}\cdot\boldsymbol{e}_X + \frac{\partial \boldsymbol{u}}{\partial X}\cdot\boldsymbol{e}_Y\end{aligned} \tag{3}$$

The dot product formulas following the familiar ones are obtained similarly to the expressions (21) in Paavola and Salonen [1]. Thus, at the local origin:

$$\begin{aligned}\varepsilon_\alpha &= \varepsilon_X = \frac{1}{h_\alpha}\frac{\partial \boldsymbol{u}}{\partial \alpha}\cdot\boldsymbol{e}_\alpha \\ \varepsilon_\beta &= \varepsilon_Y = \frac{1}{h_\beta}\frac{\partial \boldsymbol{u}}{\partial \beta}\cdot\boldsymbol{e}_\beta \\ \gamma_{\alpha\beta} &= \gamma_{XY} = \frac{1}{h_\beta}\frac{\partial \boldsymbol{u}}{\partial \beta}\cdot\boldsymbol{e}_\alpha + \frac{1}{h_\alpha}\frac{\partial \boldsymbol{u}}{\partial \alpha}\cdot\boldsymbol{e}_\beta\end{aligned} \tag{4}$$

Essentially this type of expression can be found also in Morley [4, p. 214]. The calculations proceed in the same way as, say, when evaluating the divergence in Paavola and Salonen [1] – the last form of (1) is substituted – and we just give the final results:

$$\varepsilon_\alpha = \frac{1}{h_\alpha}\left(\frac{\partial u_\alpha}{\partial \alpha} + \frac{u_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial \beta}\right)$$

$$\varepsilon_\beta = \frac{1}{h_\beta}\left(\frac{\partial u_\beta}{\partial \beta} + \frac{u_\alpha}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha}\right)$$

$$\gamma_{\alpha\beta} = \frac{h_\beta}{h_\alpha}\frac{\partial}{\partial \alpha}\left(\frac{u_\beta}{h_\beta}\right) + \frac{h_\alpha}{h_\beta}\frac{\partial}{\partial \beta}\left(\frac{u_\alpha}{h_\alpha}\right) \tag{5}$$

In the polar coordinate case:

$$\boldsymbol{u} = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta \tag{6}$$

and we have directly:

$$\varepsilon_r = \frac{1}{h_r}\frac{\partial \boldsymbol{u}}{\partial r}\cdot\boldsymbol{e}_r = \left(\frac{\partial u_r}{\partial r}\boldsymbol{e}_r + \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_\theta\right)\cdot\boldsymbol{e}_r = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_\theta = \frac{1}{h_\theta}\frac{\partial \boldsymbol{u}}{\partial \theta}\cdot\boldsymbol{e}_\theta = \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta}\boldsymbol{e}_r + u_r\boldsymbol{e}_\theta + \frac{\partial u_\theta}{\partial \theta}\boldsymbol{e}_\theta - u_\theta\boldsymbol{e}_r\right)\cdot\boldsymbol{e}_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)$$

$$\gamma_{r\theta} = \frac{1}{h_\theta}\frac{\partial \boldsymbol{u}}{\partial \theta}\cdot\boldsymbol{e}_r + \frac{1}{h_r}\frac{\partial \boldsymbol{u}}{\partial r}\cdot\boldsymbol{e}_\theta$$

$$= \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta}\boldsymbol{e}_r + u_r\boldsymbol{e}_\theta + \frac{\partial u_\theta}{\partial \theta}\boldsymbol{e}_\theta - u_\theta\boldsymbol{e}_r\right)\cdot\boldsymbol{e}_r\left(\frac{\partial u_r}{\partial r}\boldsymbol{e}_r + \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_\theta\right)\cdot\boldsymbol{e}_\theta$$

$$= \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r} \tag{7}$$

Usually [e.g. 3, p. 65], these results are obtained from some rather awkward differential geometry figures.

The large strain expressions e.g. [5, p. 435] present no theoretical difficulties:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \tag{8}$$

There are obtained first

$$\varepsilon_X = \frac{\partial \boldsymbol{u}}{\partial X}\cdot\boldsymbol{e}_X + \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{u}}{\partial X}\cdot\boldsymbol{e}_X\right)^2 + \left(\frac{\partial \boldsymbol{u}}{\partial X}\cdot\boldsymbol{e}_Y\right)^2\right]$$

$$\ldots \tag{9}$$

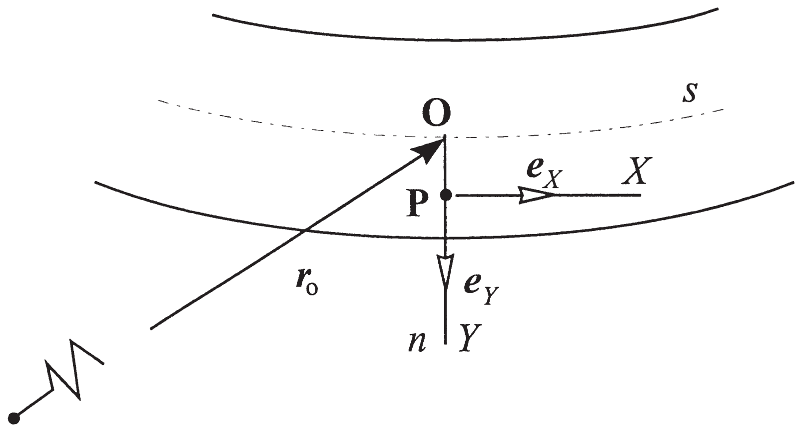

Fig. 1 *Curved beam.*

and then

$$\varepsilon_\alpha = \frac{1}{h_\alpha}\frac{\partial \boldsymbol{u}}{\partial \alpha}\cdot\boldsymbol{e}_\alpha + \frac{1}{2}\left[\left(\frac{1}{h_\alpha}\frac{\partial \boldsymbol{u}}{\partial \alpha}\cdot\boldsymbol{e}_\alpha\right)^2 + \left(\frac{1}{h_\alpha}\frac{\partial \boldsymbol{u}}{\partial \alpha}\cdot\boldsymbol{e}_\beta\right)^2\right]$$

$$\dots \tag{10}$$

Strains, curved beam

A curved plane beam is considered using the notation of Fig. 1. The curved axis of the beam is taken to be an α-coordinate line and on this line α is here associated with the arc length s. The β-coordinate lines are straight and $\beta = n$, where n is the perpendicular distance from the beam axis. The direction of coordinate n is 90° clockwise from the positive direction of s.

The purpose is to derive the expressions for the strains according to the assumptions of beam theory. In this application the method of local Cartesian coordinates proves to be very powerful.

The position vector of point P is

$$\boldsymbol{r}(s,n) = \boldsymbol{r}_0(s) + n\boldsymbol{e}_n(s) \tag{11}$$

From curve theory, $\mathrm{d}\boldsymbol{r}_0/\mathrm{d}s = \boldsymbol{e}_s$ and from the well known Frenet formulae [e.g. 6, p. 273]:

$$\frac{\mathrm{d}\boldsymbol{e}_s}{\mathrm{d}s} = -\frac{1}{R}\boldsymbol{e}_n, \quad \frac{\mathrm{d}\boldsymbol{e}_n}{\mathrm{d}s} = \frac{1}{R}\boldsymbol{e}_s \tag{12}$$

The curvature $1/R$ of the beam axis is here positive if the centre of curvature is on the negative side of the n-axis.

Differentiation of expression (11) gives thus

$$\frac{\partial \boldsymbol{r}}{\partial s}=\frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}s}+n\frac{\mathrm{d}\boldsymbol{e}_n}{\mathrm{d}s}=\boldsymbol{e}_s+n\frac{\boldsymbol{e}_s}{R}=\left(1+\frac{n}{R}\right)\boldsymbol{e}_s$$

$$\frac{\partial \boldsymbol{r}}{\partial n}=\boldsymbol{e}_n \qquad (13)$$

The scale factors are

$$h_s=\left|\frac{\partial \boldsymbol{r}}{\partial s}\right|=1+\frac{n}{R}, \quad h_n=\left|\frac{\partial \boldsymbol{r}}{\partial n}\right|=1 \qquad (14)$$

At the local origin, according to formulas (12) and (13) in Paavola and Salonen [1]:

$$\boldsymbol{e}_X=\boldsymbol{e}_s, \quad \boldsymbol{e}_Y=\boldsymbol{e}_n \qquad (15)$$

and

$$\frac{\partial}{\partial X}=\left(1+\frac{n}{R}\right)^{-1}\frac{\partial}{\partial s}, \quad \frac{\partial}{\partial Y}=\frac{\partial}{\partial n} \qquad (16)$$

We take here the small displacement assumption according to Timoshenko's beam theory [e.g. 5, p. 322], where plane sections remain plane but not necessarily perpendicular to the beam axis after deformation. This gives for point P:

$$\boldsymbol{u}(s,n)=u_s\boldsymbol{e}_s+u_n\boldsymbol{e}_n=[u(s)-n\theta(s)]\boldsymbol{e}_s(s)+v(s)\boldsymbol{e}_n(s) \qquad (17)$$

Quantities u and v are the displacement components of the origin 0 of the cross-section in the s- and n-directions, respectively, and θ is the rotation (positive clockwise) of the cross-section.

The relevant strain components in beam theory are (see expressions (3))

$$\varepsilon_s=\varepsilon_X=\frac{\partial \boldsymbol{u}}{\partial X}\cdot\boldsymbol{e}_X=\left(1+\frac{n}{R}\right)^{-1}\frac{\partial \boldsymbol{u}}{\partial s}\cdot\boldsymbol{e}_s$$

$$\gamma_{sn}=\gamma_{XY}=\frac{\partial \boldsymbol{u}}{\partial Y}\cdot\boldsymbol{e}_X+\frac{\partial u}{\partial X}\cdot\boldsymbol{e}_Y=\frac{\partial u}{\partial n}\cdot\boldsymbol{e}_s+\left(1+\frac{n}{R}\right)^{-1}\frac{\partial \boldsymbol{u}}{\partial s}\cdot\boldsymbol{e}_n \qquad (18)$$

The derivatives

$$\frac{\partial \boldsymbol{u}}{\partial s}=\left(\frac{\mathrm{d}u}{\mathrm{d}s}-n\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)\boldsymbol{e}_s+(u-n\theta)\left(-\frac{\boldsymbol{e}_n}{R}\right)+\frac{\mathrm{d}v}{\mathrm{d}s}\boldsymbol{e}_n+v\frac{\boldsymbol{e}_s}{R}$$

$$\frac{\partial \boldsymbol{u}}{\partial n}=-\theta\boldsymbol{e}_s \qquad (19)$$

and substitution in (18) gives

$$\varepsilon_s=\left(1+\frac{n}{R}\right)^{-1}\left(\frac{\mathrm{d}u}{\mathrm{d}s}-n\frac{\mathrm{d}\theta}{\mathrm{d}s}+\frac{v}{R}\right)$$

$$\gamma_{sn}=\left(1+\frac{n}{R}\right)^{-1}\left(-\frac{u-n\theta}{R}+\frac{\mathrm{d}v}{\mathrm{d}s}\right)-\theta \qquad (20)$$

To derive these exact results by some differential geometry-type figures is, in our opinion, practically impossible.

In the Bernoulli–Euler theory, it is assumed that the shear strain is zero. The latter expression in (20) can be put in the following form:

$$\gamma_{sn} = \left(1+\frac{n}{R}\right)^{-1}\left(\frac{\mathrm{d}v}{\mathrm{d}s}-\frac{u}{R}-\theta\right) \tag{21}$$

By demanding $\gamma_{sn} = 0$, we obtain the constraint:

$$\theta = \frac{\mathrm{d}v}{\mathrm{d}s}-\frac{u}{R} \tag{22}$$

Substitution of (22) into the first part of (20) gives the Bernoulli-Euler beam strain expression:

$$\begin{aligned}\varepsilon_s &= \left(1+\frac{n}{R}\right)^{-1}\left[\frac{\mathrm{d}u}{\mathrm{d}s}-n\left(\frac{\mathrm{d}^2v}{\mathrm{d}s^2}-\frac{\mathrm{d}u/\mathrm{d}s}{R}+\frac{u}{R^2}\frac{\mathrm{d}R}{\mathrm{d}s}\right)+\frac{v}{R}\right]\\ &= \frac{\mathrm{d}u}{\mathrm{d}s}+\left(1+\frac{n}{R}\right)^{-1}\left[\frac{v}{R}-n\left(\frac{u}{R^2}\frac{\mathrm{d}R}{\mathrm{d}s}+\frac{\mathrm{d}^2v}{\mathrm{d}s^2}\right)\right]\end{aligned} \tag{23}$$

The expressions obtained can be approximated for shallow beams by developing them in truncated power series in n/R.

General equilibrium equations

The general form of stress equilibrium equations for a continuum is given by Malvern [7, p. 215] as:

$$\mathrm{div}\boldsymbol{\sigma}+\boldsymbol{f}=\boldsymbol{0} \tag{24}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\boldsymbol{f}$ the body force vector intensity (per volume). It should be noted that we consider the equations in this article to be described in Lagrangian (material) coordinates, as is usual in solid mechanics [7, p. 138]. Equation (24) is in fact exact only in Eulerian (spatial) coordinates but, as is well known, can be used as a valid approximation as such in the small displacement theory considering the coordinates as material ones. Using dyadic representation we have in Cartesian coordinates:

$$\mathrm{div}\boldsymbol{\sigma} = \left(\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}\right)\cdot\left(\sigma_x\boldsymbol{ii}+\tau_{xy}\boldsymbol{ij}+\tau_{yx}\boldsymbol{ji}+\sigma_y\boldsymbol{jj}\right) \tag{25}$$

where σ_x, σ_y and $\tau_{xy} = \tau_{yx}$ are the stress components. This transforms to:

$$\mathrm{div}\boldsymbol{\sigma} = \left(\boldsymbol{e}_\alpha\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}+\boldsymbol{e}_\beta\frac{1}{h_\beta}\frac{\partial}{\partial\beta}\right)\cdot\left(\sigma_\alpha\boldsymbol{e}_\alpha\boldsymbol{e}_\alpha+\tau_{\alpha\beta}\boldsymbol{e}_\alpha\boldsymbol{e}_\beta+\tau_{\beta\alpha}\boldsymbol{e}_\beta\boldsymbol{e}_\alpha+\sigma_\beta\boldsymbol{e}_\beta\boldsymbol{e}_\beta\right) \tag{26}$$

for curvilinear coordinates. A rather long manipulation finally gives the equations:

$$\frac{1}{h_\alpha h_\beta}\left[\frac{\partial}{\partial\alpha}\left(h_\beta\sigma_\alpha\right)+\frac{\partial}{\partial\beta}\left(h_\alpha\tau_{\beta\alpha}\right)+\frac{\partial h_\alpha}{\partial\beta}\tau_{\alpha\beta}-\frac{\partial h_\beta}{\partial\alpha}\sigma_\beta\right]+f_\alpha=0$$
$$\frac{1}{h_\beta h_\alpha}\left[\frac{\partial}{\partial\beta}\left(h_\alpha\sigma_\beta\right)+\frac{\partial}{\partial\alpha}\left(h_\beta\tau_{\alpha\beta}\right)+\frac{\partial h_\beta}{\partial\alpha}\tau_{\beta\alpha}-\frac{\partial h_\alpha}{\partial\beta}\sigma_\alpha\right]+f_\beta=0 \qquad (27)$$

Equation (24), however, is clearly not a suitable starting point for basic courses. The equation itself is probably not familiar and the manipulations needed are rather tedious. The principle of virtual work gives an alternative way to produce the equilibrium equations.

The principle of virtual work has a central role in teaching structural mechanics. It can be employed in analytical applications but it also forms the starting point for numerical methods, especially the finite element method. It is thus a unifying principle and so should be clearly understood by the student. It is well known that the method can be used to derive equilibrium equations in a systematic way [e.g. 8]. Here this idea is employed in connection with the method of local Cartesian coordinates. The resulting general formulas are unavoidably rather complicated. They are given here for completeness but, again, it is stressed that for teaching purposes there is usually no need to operate with these general expressions, as in specific applications the formulas needed can be obtained more easily directly.

The principle of virtual work can be expressed [e.g. 9, p. 230] as:

$$\delta W^i+\delta W^e=0 \qquad (28)$$

Here, for a two-dimensional continuum (with small strains), the virtual work of internal forces is:

$$\delta W^i=-\int_A\left(\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\tau_{xy}\delta\gamma_{xy}\right)\mathrm{d}A \qquad (29)$$

and the virtual work of external forces, correspondingly, is:

$$\delta W^e=\int_A\left(f_x\delta u+f_y\delta v\right)\mathrm{d}A+\int_{s_t}\left(t_x\delta u+t_y\delta v\right)\mathrm{d}s \qquad (30)$$

The virtual strains and displacements are related by (see (2) and use the commutative law between variation and differentiation):

$$\delta\varepsilon_x=\frac{\partial\delta u}{\partial x}$$
$$\delta\varepsilon_y=\frac{\partial\delta v}{\partial y}$$
$$\delta\gamma_{xy}=\frac{\partial\delta u}{\partial y}+\frac{\partial\delta v}{\partial x} \qquad (31)$$

Boundary s_t is that part of the total boundary where the traction components t_x and t_y are given.

The integrands in (29) and (30) are, because of their physical meaning, invariants with respect to coordinate transformations. Thus we can write immediately the relations:

$$\begin{aligned}
&\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\tau_{xy}\delta\varepsilon_{xy}=\sigma_X\delta\varepsilon_X+\sigma_Y\delta\varepsilon_Y+\tau_{XY}\delta\gamma_{XY}\\
&=\sigma_\alpha\delta\varepsilon_\alpha+\sigma_\beta\delta\varepsilon_\beta+\tau_{\alpha\beta}\delta\gamma_{\alpha\beta}\\
&f_x\delta u+f_y\delta v=f_X\delta u_X+f_Y\delta u_Y=f_\alpha\delta u_\alpha+f_\beta\delta u_\beta\\
&t_x\delta u+t_y\delta v=t_X\delta u_X+t_Y\delta u_Y=t_\alpha\delta u_\alpha+t_\beta\delta u_\beta
\end{aligned}\tag{32}$$

The principle of virtual work (28) in curvilinear coordinates now obtains the form:

$$\begin{aligned}
&-\int_A\left(\sigma_\alpha\delta\varepsilon_\alpha+\sigma_\beta\delta\varepsilon_\beta+\tau_{\alpha\beta}\delta\gamma_{\alpha\beta}\right)\mathrm{d}A+\int_A\left(f_\alpha\delta u_\alpha+f_\beta\delta u_\beta\right)\mathrm{d}A\\
&+\int_{s_t}\left(t_\alpha\delta u_\alpha+t_\beta\delta u_\beta\right)\mathrm{d}s=0
\end{aligned}\tag{33}$$

where, from (5):

$$\begin{aligned}
\delta\varepsilon_\alpha&=\frac{1}{h_\alpha}\left(\frac{\partial\delta u_\alpha}{\partial\alpha}+\frac{\delta u_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial\beta}\right)\\
\delta\varepsilon_\beta&=\frac{1}{h_\beta}\left(\frac{\partial\delta u_\beta}{\partial\beta}+\frac{\delta u_\alpha}{h_\alpha}\frac{\partial h_\beta}{\partial\beta}\right)\\
\delta\gamma_{\alpha\beta}&=\frac{h_\beta}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\delta u_\beta}{h_\beta}\right)+\frac{h_\alpha}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{\delta u_\alpha}{h_\alpha}\right)
\end{aligned}\tag{34}$$

The expression for the internal virtual work is manipulated further, with $\mathrm{d}A=h_\alpha h_\beta\mathrm{d}\alpha\mathrm{d}\beta$:

$$\begin{aligned}
\delta W^i=&-\int_{\alpha,\beta}\left\{\sigma_\alpha h_\beta\left(\frac{\partial\delta u_\alpha}{\partial\alpha}+\frac{\delta u_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial\beta}\right)+\sigma_\beta h_\alpha\left(\frac{\partial\delta u_\beta}{\partial\beta}+\frac{\delta u_\alpha}{h_\alpha}\frac{\partial h_\beta}{\partial\alpha}\right)\right.\\
&\left.+\tau_{\alpha\beta}\left[h_\beta^2\frac{\partial}{\partial\alpha}\left(\frac{\delta u_\beta}{h_\beta}\right)+h_\alpha^2\frac{\partial}{\partial\beta}\left(\frac{\delta u_\alpha}{h_\alpha}\right)\right]\right\}\mathrm{d}\alpha\mathrm{d}\beta\\
=&\int_{\alpha,\beta}\left[\frac{\partial\left(h_\beta\sigma_\alpha\right)}{\partial\alpha}\delta u_\alpha+\frac{\partial\left(h_\alpha\sigma_\beta\right)}{\partial\beta}\delta u_\beta+\frac{\partial\left(h_\beta^2\tau_{\alpha\beta}\right)}{\partial\alpha}\frac{\delta u_\beta}{h_\beta}+\frac{\partial\left(h_\alpha^2\tau_{\alpha\beta}\right)}{\partial\beta}\frac{\delta u_\alpha}{h_\alpha}\right.\\
&\left.-\frac{\partial h_\alpha}{\partial\beta}\sigma_\alpha\delta u_\beta-\frac{\partial h_\beta}{\partial\alpha}\sigma_\beta\delta u_\alpha\right]\mathrm{d}\alpha\mathrm{d}\beta\\
&-\int_s\left(\frac{n_\alpha}{h_\beta}\sigma_\alpha h_\beta\delta u_\alpha+\frac{n_\beta}{h_\alpha}\sigma_\beta h_\alpha\delta u_\beta+\frac{n_\alpha}{h_\beta}\tau_{\alpha\beta}h_\beta^2\frac{\delta u_\beta}{h_\beta}+\frac{n_\beta}{h_\alpha}\tau_{\alpha\beta}h_\alpha^2\frac{\delta u_\alpha}{h_\alpha}\right)\mathrm{d}s
\end{aligned}\tag{35}$$

Integration by parts formulas (42) in Paavola and Salonen [1] have been applied to free the derivatives operating on the virtual displacement components. Equation (33) now becomes:

$$\int_{\alpha,\beta}\left\{\left[\frac{\partial}{\partial\alpha}\left(h_\beta\sigma_\alpha\right)+\frac{1}{h_\alpha}\frac{\partial}{\partial\beta}\left(h_\alpha^2\tau_{\alpha\beta}\right)-\frac{\partial h_\beta}{\partial\alpha}\sigma_\beta+h_\alpha h_\beta f_\alpha\right]\delta u_\alpha+\right.$$
$$\left.\left[\frac{\partial}{\partial\beta}\left(h_\alpha\sigma_\beta\right)+\frac{1}{h_\beta}\frac{\partial}{\partial\alpha}\left(h_\beta^2\tau_{\alpha\beta}\right)-\frac{\partial h_\alpha}{\partial\beta}\sigma_\alpha+h_\alpha h_\beta f_\beta\right]\delta u_\beta\right\}\mathrm{d}\alpha\mathrm{d}\beta+$$
$$\int_{s_t}\left\{\left[t_\alpha-n_\alpha\sigma_\alpha-n_\beta\tau_{\alpha\beta}\right]\delta u_\alpha+\left[t_\beta-n_\beta\sigma_\beta-n_\alpha\tau_{\alpha\beta}\right]\delta u_\beta\right\}\mathrm{d}s=0 \tag{36}$$

Kinematically admissible virtual displacements have been used, so that $\delta u_\alpha = 0$, $\delta u_\beta = 0$ on the part $s\backslash s_t$ of the boundary where the displacements are given. As the virtual displacements are arbitrary, the terms in brackets in (36) must separately vanish. This gives the stress equilibrium equations (27) – in a slightly modified form – and, in addition, on the boundary, the traction–stress relations:

$$t_\alpha = n_\alpha\sigma_\alpha + n_\beta\tau_{\alpha\beta}$$
$$t_\beta = n_\beta\sigma_\beta + n_\alpha\tau_{\alpha\beta} \tag{37}$$

The equivalents of (33) and (34) in polar coordinates are:

$$-\int_A\left(\sigma_r\delta\varepsilon_r+\sigma_\theta\delta\varepsilon_\theta+\tau_{r\theta}\delta\gamma_{r\theta}\right)\mathrm{d}A$$
$$+\int_A\left(f_r\delta u_r+f_\theta\delta u_\theta\right)\mathrm{d}A+\int_{s_t}\left(t_r\delta u_r+t_\theta\delta u_\theta\right)\mathrm{d}s=0 \tag{38}$$

$$\delta\varepsilon_r=\frac{\partial\delta u_r}{\partial r}$$
$$\delta\varepsilon_\theta=\frac{1}{r}\left(\delta u_r+\frac{\partial\delta u_r}{\partial\theta}\right)$$
$$\delta\gamma_{r\theta}=\frac{1}{r}\left(\frac{\partial\delta u_r}{\partial\theta}-\delta u_\theta\right)+\frac{\partial\delta u_\theta}{\partial r} \tag{39}$$

Equation (38) obtains with $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$ first in the form:

$$-\int_{r,\theta}\left[r\sigma_r\frac{\partial\delta u_r}{\partial r}+\sigma_\theta\left(\delta u_r+\frac{\partial\delta u_\theta}{\partial\theta}\right)+\tau_{r\theta}\left(\frac{\partial\delta u_r}{\partial\theta}-\delta u_\theta\right)+r\tau_{r\theta}\frac{\partial\delta u_\theta}{\partial r}\right]\mathrm{d}r\mathrm{d}\theta$$
$$+\int_{r,\theta}\left(rf_r\delta u_r+rf_\theta\delta u_\theta\right)\mathrm{d}r\mathrm{d}\theta+\int_{s_t}\left(t_r\delta u_r+t_\theta\delta u_\theta\right)\mathrm{d}s=0 \tag{40}$$

Application of formulas (44) in Paavola and Salonen [1] gives, further:

$$\int_{r,\theta}\left\{\left[\frac{\partial(r\sigma_r)}{\partial r}-\sigma_\theta+\frac{\partial\tau_{r\theta}}{\partial\theta}+rf_r\right]\delta u_r+\left[\frac{\partial\sigma_\theta}{\partial\theta}+\tau_{r\theta}+\frac{\partial(r\tau_{r\theta})}{\partial r}+rf_\theta\right]\delta u_\theta\right\}\mathrm{d}r\mathrm{d}\theta+$$
$$\int_{s_t}\{[t_r-n_r\sigma_r-n_\theta\tau_{r\theta}]\delta u_r+[t_\theta-n_\theta\sigma_\theta-n_r\tau_{r\theta}]\delta u_\theta\}\mathrm{d}s=0 \qquad (41)$$

The equilibrium equations are thus – after some minor development:

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}+f_r=0$$
$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{2\tau_{r\theta}}{r}+f_\theta=0 \qquad (42)$$

$$t_r=n_r\sigma_r+n_\theta\tau_{r\theta}$$
$$t_\theta=n_\theta+n_r\tau_{r\theta} \qquad (43)$$

Concluding remarks

The equilibrium equations for the curved beam employing stress resultants can be obtained similarly as above, using the principle of virtual work, but for lack of space the derivations are not given here. Further, it is not difficult to include the effect of inertia forces to derive the corresponding equations in dynamics. In solid mechanics, the curved beam is useful as a didactic step towards full shell equations, which can be formulated quite nicely with the help of the method of local Cartesian coordinates [10].

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