Control theory, delays and driving an automobile

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Abstract A simple model of the driver–automobile system is developed and several reasonable candidates for the transfer function of the human 'controller' are studied. The model is used to examine the controller candidates. The complete system is analysed both analytically and through simulations. It is found that a delay followed by a phase-lead controller is a reasonable choice for the transfer function that the human 'controller' implements. The model developed is compared with a more realistic model and is seen to be a reasonable approximation of the realistic model at low frequencies. Reaction time is shown to be a critical parameter in understanding the dynamics of the driver–automobile system.

Keywords mathematical modelling; delays; reaction time; automobiles

Notation

$x(t)$	The distance from the rear tyre to the side of the lane
$y(t)$	The distance from the front tyre to the side of the lane
$\theta(t)$	The automobile's angle with respect to the lane
$\theta_s(t)$	The total angle through which the steering wheel has been turned at time t
$\theta_w(t)$	The angle of the automobile's front tyres with respect to the centre-line
K_s	The (fixed) ratio of $\theta_w(t)$ to $\theta_s(t)$
l	The length of the automobile (as measured by the distance from the centre of the front tyre to the centre of the rear tyre)
τ	The characteristic open-loop delay of the automobile–driver system
$r(t)$	The position the driver would like to be at
c	The pure-gain portion of the transfer function implemented by the driver
$T(s)$	The frequency-dependent portion of the transfer function implemented by the driver
$G_c(s)$	The transfer function that is being implemented by the driver. (This does, however, include all pure delays.) $G_c(s) = cT(s)e^{-\tau s}$
$G_p(s)$	The automobile's transfer function
v	The speed of the automobile
$e(t)$	The difference between the desired and the actual position of the automobile. $e(t) = r(t) - x(t)$
$\varphi(t)$	The combined phase of the compensator and the delay
ω_{mag}	If $\omega \geq \omega_{mag}$, then $\lvert G_c(j\omega)G_p(j\omega)\rvert < 1$
ω_{phase}	If $\omega \leq \omega_{phase}$, then $\angle(G_c(j\omega)G_p(j\omega)) > -\pi$

Introduction

In this article, a model for the driver–automobile system is developed. The model is examined, as are the effects of the control strategy and the reaction time of the driver. The model is compared with the more realistic model developed by Hess and Modjtahedzadeh [1] and is seen to be a reasonable approximation of their model at low frequencies.

Let the 'input' to the driver be the difference between where the car is, $x(t)$, and where the car ought to be, $r(t)$. Assume that the driver turns the car's steering wheel an angle $\theta_s(t)$ in such a way that the change in the car's direction acts to reduce the error that the driver *anticipates* seeing in the near future. A set of control strategies is developed that model such behaviour; the conditions under which the strategies lead to a stable system are derived, and the properties of the system are examined analytically and using Matlab and Simulink. It is shown that reaction time plays a very important role in determining whether a given system is stable and whether a given controller tends to cause the automobile to 'hunt' for its lane.

Pedagogic content

This article presents a method for modelling the driver–automobile system using some simple mathematical modelling, some knowledge about how people process data, and control theory. A student ought to be able to understand the contents of this article at the end of a one-semester course on control theory. In this article use is made of:

- mathematics, to develop a simple model of a car;
- the Nyquist criterion, to determine conditions under which the proposed systems will be stable or unstable;
- the root-locus plot, to examine the conditions under which the systems considered become underdamped;
- the Padé approximation, to make it possible to use the root-locus technique to examine systems with delays;
- some calculus.

By putting these items together it is possible to understand many of the everyday phenomena that we see as automobile users. This material is a good foundation for a last lecture (or series of lectures) in a course on control theory. For more background information about the driver–automobile system see the references in [1].

A set of transparencies – in PDF form – that presents many of the ideas developed in this paper is available from the author upon request. A lecture based on these transparencies has been used as the final lecture in a course on control theory. As a result of the lecture a new appreciation of mathematical modelling and of the applications of control theory was gained by the students.

Modelling the effects of steering

When a car's steering wheel is turned an angle of θ_s, the front wheels of the car turn $\theta_w = K_s\theta_s$ where K_s is the amplification of the system that causes the steering wheel

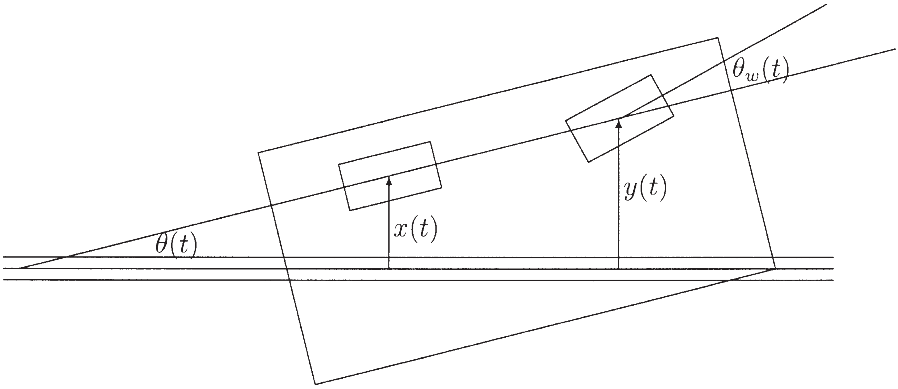

Fig. 1 *The automobile's state. Here* θ(t) *is the automobile's angle with respect to the lane;* $\theta_w(t)$ *is the angle of the front tyre relative to the automobile's centre-line;* x(t) *is the distance from the rear tyre to the lane, and* y(t) *is the distance from the front tyre to the lane. In the figure the automobile is travelling up and to the right.*

to determine the tyres' position. Assuming that the car is already at an angle θ relative to the road (and considering only one pair of wheels) the automobile's state is shown is Fig. 1.

Assuming that the automobile is moving at a constant speed, v, and that each wheel is moving at the automobile's speed, it is found that [2]:

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = \sin(\theta(t) + \theta_w(t))v$$

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \sin(\theta(t))v$$

$$\theta(t) = \sin^{-1}\left(\frac{y(t) - x(t)}{l}\right)$$

where l is the automobile's length. Assuming that the angles involved are all small, these equations can be approximated by:

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = (\theta(t) + \theta_{\mathrm{w}}(t))v \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \theta(t)v \tag{2}$$

$$\theta(t) = \frac{y(t) - x(t)}{l} \tag{3}$$

Differentiating equation 3 and using equations 1 and 2, it is found that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) = \frac{\theta_{\mathrm{w}}(t)v}{l} \tag{4}$$

Now, differentiating equations 1 and 2 and using equation 4, it is found that:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) = (\theta'(t) + \theta'_{\mathrm{w}}(t))v = \frac{\theta_{\mathrm{w}}(t)v^2}{l} + \theta'_{\mathrm{w}}(t)v$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) = (\theta_{\mathrm{w}}(t)v)v/l = \theta_{\mathrm{w}}(t)v^2/l$$

It is found [2] that the relationship between the angle of the steering wheel and the lateral ('crosswise') position of the automobile, $x(t)$, is:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) = \frac{K_{\mathrm{s}}\theta_{\mathrm{s}}(t)v^2}{l} \tag{5}$$

The controller – in this case the driver – must decide how to drive based on what is observed. Because it takes a person some time to make decisions and it takes the driver–automobile system some time to put the decisions into effect, we assume that one part of the transfer function of the controller is a delay of τ seconds – in the Laplace transform domain $e^{-\tau s}$. A reasonable number for τ is about one second [3].

The input to the whole 'system' is $r(t)$, which is where the driver *would currently like to be* in the lane. If an obstacle suddenly appears in front of the driver, then the input changes by jumping to another value; that is, the system sees a step input. The 'input' to the driver is the difference between the desired position, $r(t)$, and the car's actual position, $x(t)$. The driver reacts to the input by changing the angle of the steering wheel. Thus far, all that has been seen is that part of the transfer function of the driver is $e^{-\tau s}$. The rest of the transfer function is written as $cT(s)$, where c is a pure amplification and $T(s)$ is the rest of the frequency-dependent portion of the driver's transfer function. It is found that:

$$\theta_{\mathrm{s}}(s) = (R(s) - X(s))cT(s)e^{-\tau s}$$

where capital letters are used to denote Laplace transforms of functions of time. From equation 5 it is found that:

$$X(s) = \frac{K_{\mathrm{s}}v^2}{ls^2}\theta_{\mathrm{s}}(s).$$

A similar transfer function is given to describe the vehicle dynamics of a car travelling at 50 km/h in [1]. At low frequencies the transfer function given there is:

$$T(s) = \frac{11.54}{s^2}$$

The transfer function of the person – the controller – is:

$$G_{\mathrm{c}}(s) = cT(s)e^{-\tau s}$$

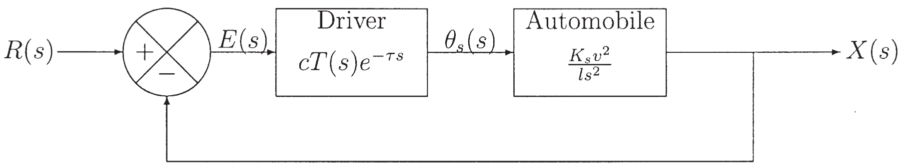

Fig. 2 *The block diagram of the driver–car system.*

and the transfer function of the car – the plant – is:

$$G_p(s) = \frac{K_s v^2}{ls^2}$$

The system that has been described – the system shown in Fig. 2 – is analysed in what follows.

The controllers

There are several ways in which a person might process the error signal, $e(t) \equiv r(t) - x(t)$, before using it to change the angle of the steering wheel, $\theta_s(t)$. The least likely but most easily considered method of processing is to differentiate the data. Under this condition $T(s) = s$ and the controller is a derivative controller. Using such a controller makes the system first order – and then the system cannot track a ramp input. Additionally, if a derivative controller is used, then in the product of the driver's transfer function and the automobile's transfer function an unstable pole (a pole with a non-negative real part) of the automobile's transfer function, $1/s$, is cancelled by a zero of the driver's transfer function, s. That is:

$$cT(s)e^{-\tau s} \cdot \frac{K_s v^2}{ls^2} = ce^{-\tau s} s \cdot \frac{1}{s} \frac{K_s v^2}{ls} = ce^{-\tau s} \frac{K_s v^2}{ls}$$

Such cancellations cause the system to be internally unstable. A system is internally stable if a bounded signal that is injected at any point in the system – that is added to the rest of the signals arriving at that point – cannot cause the output at any other point in the system to become unbounded [4]. In order for a system to be internally stable, it is necessary and sufficient for the system to be stable according to the Nyquist criterion and that it have no pole-zero cancellations involving unstable poles [4, p. 32, theorem 2]. A system that is not internally stable is, generally, unusable. Its response to bounded inputs may be fine as long as no disturbances are present internally, as is the case here, but when the 'wrong' disturbances are injected at some internal point some of the signals in the system grow without bound. As shall be seen, there are better controllers that lead to systems that can track ramp inputs, that are internally stable, and whose transfer function can usefully be approximated by the transfer function of a derivative controller.

The other controllers considered involve extrapolating what the error should be at some point in the future and trying to correct for that error. To extrapolate from the current value of the error in a linear fashion, one makes use of the fact that:

$$e(t+\Delta t)\approx e(t)+e'(t)\Delta t$$

To implement such an extrapolation, one chooses $T(s) = 1 + s\Delta t$. This is a proportional-derivative (PD) controller.

The problem with using such a controller directly is that it involves differentiation and hence its gain is unbounded; at high frequencies the gain tends to infinity. A somewhat different way to extrapolate is to use a function that is similar to the above when $s = j\omega$ is small – which is when the above estimate is accurate – and is bounded at higher frequencies. A simple example of such a function is a phase-lead compensator [5]:

$$T(s)=\frac{s/\omega_o+1}{s/\omega_p+1}\quad \omega_p>\omega_o$$

For small values of s this can be approximated as:

$$T(s)=\frac{s/\omega_o+1}{s/\omega_p+1}\approx\left(1+s/\omega_o\right)\left(1-s/\omega_p\right)\approx\left(1+s/\omega_o-s/\omega_p\right)$$

Thus, for small s, the two methods are approximately the same and:

$$\Delta t=\frac{1}{\omega_o}-\frac{1}{\omega_p}=\frac{\omega_p-\omega_o}{\omega_p\omega_o} \tag{6}$$

Additionally, one can recover the PD controller from the phase-lead controller by allowing $\omega_p \to \infty$.

The derivative controller

In this section the driver–automobile system is examined when $T(s) = s$. Though this transfer function is not terribly useful as it stands – as has been shown, it leads to a system that is not internally stable – there are practical transfer functions that can be approximated by such a transfer function, as will be seen in the next section (‘The phase-lead controller’).

The Nyquist plot analysis

As with all control systems, the first question that must be asked is, ‘Under what conditions is this system stable?’ As with all systems whose transfer functions are low pass but that include a delay, one way of answering this question is to analyse the Nyquist plot of the system. If −1 is never encircled by the Nyquist plot, then a low-pass system with a delay whose constituent parts have no poles in the right half-plane is stable.

In order to determine whether the Nyquist plot encircles −1 it is necessary to understand the behaviour of the function:

$$G_c(j\omega)G_p(j\omega)=\frac{K_s c v^2 e^{-\tau j\omega}}{lj\omega}$$

The absolute value of this function is:

$$\left|G_c(j\omega)G_p(j\omega)\right| = K_s cv^2 \frac{1}{l\omega}$$

This function is monotonically decreasing in the interval $\omega \in (0, \infty)$. Additionally, the angle of the function decreases monotonically from $-90°$ to $-\infty°$ in the same interval. To see whether the Nyquist plot encircles −1 under these conditions, it is sufficient to check whether, at the first intersection with the negative real axis, $G_c(j\omega)G_p(j\omega)$ is less than or equal to −1.

At the division by j furnishes us with a constant $-90°$ shift, it is found that the first time that the function is negative is when $\tau\omega = \pi/2$, that is , when $\omega_0 = \frac{\pi}{2\tau}$. At that point:

$$G_c(j\omega_0)G_p(j\omega_0) = -\frac{2K_s cv^2 \tau}{l\pi}$$

As this is the largest negative value that $G_c(j\omega)G_p(j\omega)$ can take, the system is stable until this value passes −1. We find that the condition for stability is:

$$\frac{2K_s cv^2 \tau}{l\pi} < 1$$

or that:

$$\frac{K_s cv^2 \tau}{l} < \pi/2 = 1.5708$$

From this we conclude that:

- A given person must be gentler with the steering wheel (c must be smaller) as the automobile's speed, v, increases. Otherwise the person will lose control of the automobile.
- A person whose reflexes are bad – whose reaction time is slow (for whom τ is large) – must be gentler with the steering wheel (c must be smaller). Otherwise the person will lose control of the automobile.

The root-locus analysis

Though one cannot use root-locus analysis directly on a transfer function that includes a delay, after approximating the delay – after approximating $e^{-\tau s}$ – by a rational function, one can use root-locus techniques. Making use of the fact that our compensator is a differentiator, of the form of $G_p(s)$, and of the Padé approximation [6, p. 191]:

$$e^{-\tau s} \approx \frac{1-\frac{\tau}{2}s}{1+\frac{\tau}{2}s} \quad |\tau s| << 1$$

it is found that:

$$G_c(s)G_p(s) \propto \frac{1}{s}\frac{1-\frac{\tau}{2}s}{1+\frac{\tau}{2}s}$$

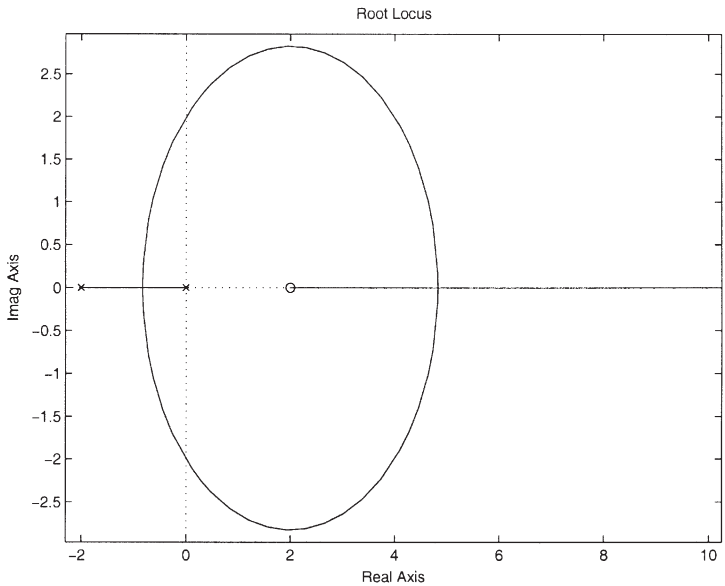

Fig. 3 *The root-locus plot of the approximation.*

(The pure-gain terms are irrelevant at this point, as they do not change the root-locus plot.) A picture of the root-locus diagram when $\tau = 1$ is given in Fig. 3. Note that because $G_c(s)G_p(s)$ is *negative* for large s, there are branches of the root-locus at every point on the real axis that is to the left of an *even* number of poles and zeros of the transfer function.

Fig. 3 shows that when the gain in the system crosses a threshold, the system starts oscillating. When the gain is still higher, the system becomes unstable.

To find the point at which the system is expected to start behaving like an under-damped one, one must calculate the zeros of the derivative of $G_c(s)G_p(s)$ [5, p. 223]. We find that:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(G_c(s)G_p(s)\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{K_s c v^2}{ls}\frac{1-\frac{\tau}{2}s}{1+\frac{\tau}{2}s}\right\} = \frac{K_s c v^2}{l}\frac{\tau^2 s^2 - 4\tau s - 4}{\left(2s+\tau s^2\right)^2}$$

The zeros of this equation are:

$$s_{\pm} = \frac{2}{\tau}(1 \pm \sqrt{2})$$

In order for these values to correspond to poles, they must satisfy:

$$G_c(s_{\pm})G_p(s_{\pm}) = -1$$

It is found that the condition is:

$$\frac{K_s c v^2}{l_\tau^2(1\pm\sqrt{2})}\frac{1-(1\pm\sqrt{2})}{1+(1\pm\sqrt{2})}=\frac{K_s c v^2\tau}{l2(1\pm\sqrt{2})}\frac{1-(1\pm\sqrt{2})}{1+(1\pm\sqrt{2})}=-1$$

As s_+ corresponds to the point at which two *unstable* branches of the root-locus coalesce, the point of interest to us satisfies:

$$\frac{K_s c v^2\tau}{l2(1-\sqrt{2})}\frac{1-(1-\sqrt{2})}{1+(1-\sqrt{2})}=-2.9142\frac{K_s c v^2\tau}{l}=-1$$

We find that as long as:

$$\frac{K_s c v^2\tau}{l}<\frac{-1}{-2.9142}=0.3431$$

there should not be any overshoot in the output of the system and the system is as fast as it can get.

The phase-lead controller

Let us consider the system of Fig. 2 when:

$$T(s)=\frac{s/\omega_o+1}{s/\omega_p+1}\quad \omega_p>\omega_o$$

This system has finite gain at all frequencies and is therefore more reasonable than the pure differentiator. It is easy to see that:

$$|T(j\omega)|^2=\frac{(\omega\omega_p)^2+\omega_o^2\omega_p^2}{(\omega\omega_o)^2+\omega_o^2\omega_p^2}$$

Differentiating with respect to ω^2 and recalling that in a phase-lead controller $\omega_p > \omega_o$, it is found that:

$$\begin{aligned}\frac{d}{d\omega^2}|T(j\omega)|^2&=\frac{\omega_p^2\left((\omega\omega_o)^2+\omega_o^2\omega_p^2\right)-\omega_o^2\left((\omega\omega_p)^2+\omega_o^2\omega_p^2\right)}{\left((\omega\omega_o)^2+\omega_o^2\omega_p^2\right)^2}\\&=\frac{\omega_o^2\omega_p^4-\omega_o^4\omega_p^2}{\left((\omega\omega_o)^2+\omega_o^2\omega_p^2\right)^2}\\&=\omega_o^2\omega_p^2\frac{\omega_p^2-\omega_o^2}{\left((\omega\omega_o)^2+\omega_o^2\omega_p^2\right)^2}\\&>0\end{aligned}$$

Thus, the magnitude of the transfer function is largest as $\omega\to\infty$. In this case it is clear that this value is just:

$$\lim_{\omega\to\infty}|T(j\omega)|=\frac{\omega_p}{\omega_o}$$

The Nyquist analysis

The stability of the system will now be determined using the standard Nyquist analysis. As with all low-pass systems that include a delay, it is sufficient to examine the properties of $G_c(j\omega)G_p(j\omega)$ for $\omega > 0$. The phase of $G_c(j\omega)G_p(j\omega)$ is just:

$$\angle G_c(j\omega)G_p(j\omega) = -\tan^{-1}(\omega/\omega_p) + \tan^{-1}(\omega/\omega_o) - \pi - \tau\omega$$

Clearly, the phase will be less than $-\pi = -180°$ if:

$$\varphi(\omega) \equiv -\tau\omega - \tan^{-1}(\omega/\omega_p) + \tan^{-1}(\omega/\omega_o) < 0$$

It is clear that $\varphi(\omega)$ is just the phase of the compensator and the delay.

In order to determine the stability of the driver–automobile system, it is important to understand the behaviour of $\varphi(\omega)$. Note that $\varphi(0) = 0$. Let us calculate $\varphi'(\omega)$. It is found that:

$$\varphi'(\omega) = -\tau - \frac{\omega_p^2}{\omega^2 + \omega_p^2}\frac{1}{\omega_p} + \frac{\omega_o^2}{\omega^2 + \omega_o^2}\frac{1}{\omega_o}$$

Considering the function:

$$f(x) = \frac{x}{a+x} \quad x, a > 0$$

it is seen that:

$$f'(x) = \frac{(a+x) - x}{(a+x)^2} = \frac{a}{(a+x)^2} > 0$$

Thus, $f(x)$ is increasing and as $\omega_p > \omega_o$ we find that:

$$\varphi'(\omega) \le -\tau + \frac{\omega_p^2}{\omega^2 + \omega_p^2}\left(\frac{1}{\omega_o} - \frac{1}{\omega_p}\right)$$

Because $f(x)$ increases from 0 to 1 as x increases from 0 to ∞, we find that we can further overestimate $\varphi'(\omega)$ by:

$$\varphi'(\omega) \le -\tau + \left(\frac{1}{\omega_o} - \frac{1}{\omega_p}\right)$$

We find that if:

$$\frac{1}{\omega_o} - \frac{1}{\omega_p} < \tau \tag{7}$$

then $\varphi(\omega)$ starts at zero and is *decreasing*. Moreover, like the delay which is one of its components, it decreases toward $-\infty$.

When the condition in equation 7 is met, the Nyquist plot of $G_c(s)G_p(s)$ is similar to the plot in Fig. 4. As usual with Nyquist plots produced by Matlab, the circle at infinity caused by the poles at the origin is not plotted. In our case, as there are two poles at the origin, there is one full circle that is traced out in the counter-clockwise direction. The circle starts on the upper left-hand part of the Nyquist plot, completes

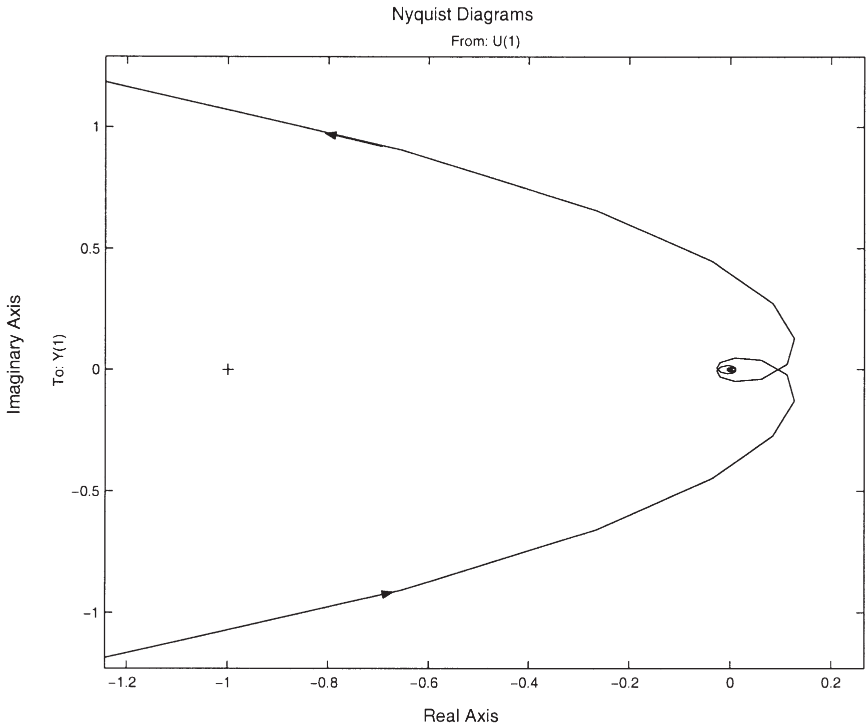

Fig. 4 *The Nyquist plot of a typical unstable system.*

a full circle and ends on the lower left-hand part of the plot. Clearly −1 is encircled (twice) and the system is not stable.

From equation 7 it is clear that a necessary condition for stability is that:

$$\frac{1}{\omega_o} - \frac{1}{\omega_p} \geq \tau$$

As equation 6 shows, this condition is equivalent to saying that the controller (driver) must predict far enough into the future to compensate for the delay inherent in the person–automobile system.

Having given conditions under which the driver–automobile system is unstable, conditions under which the system is stable are derived. It is clear that:

$$|G_c(j\omega)G_p(j\omega)| < c\frac{K_s v^2}{l}\frac{\omega_p}{\omega_o}\frac{1}{\omega^2} \quad \omega > 0$$

The magnitude is less than one if:

$$\omega > \omega_{mag} \equiv v\sqrt{c\frac{K_s}{l}\frac{\omega_p}{\omega_o}}$$

where ω_{mag} is the frequency for which the value of the overestimate of the magnitude is equal to 1.

Having examined the magnitude, the phase must now be examined. The Taylor series expansion for $\tan^{-1}(\theta)$ is:

$$\tan^{-1}(\theta) = \theta - \theta^3/3 + \theta^5/5 + \cdots + (-1)^{n+1}\theta^{2n-1}/(2n-1) + \cdots$$

When $\theta > 0$ this is an alternating series. From the properties of alternating series, it is known that:

$$\theta > \tan^{-1}(\theta) > \theta - \frac{\theta^3}{3} \quad \theta > 0$$

It is certain that $\angle G_{\mathrm{c}}(j\omega)G_{\mathrm{p}}(j\omega) > -\pi$ as long as:

$$-\tau\omega - \frac{\omega}{\omega_{\mathrm{p}}} + \frac{\omega}{\omega_{\mathrm{o}}} - \frac{\omega^3}{3\omega_{\mathrm{o}}^3} > 0$$

that is, as long as:

$$\omega < \omega_{\mathrm{phase}} \equiv \sqrt{3}\omega_{\mathrm{o}}^{3/2}\sqrt{\frac{1}{\omega_{\mathrm{o}}} - \frac{1}{\omega_{\mathrm{p}}} - \tau} > 0$$

Clearly, as long as $\omega_{\mathrm{phase}} > \omega_{\mathrm{mag}}$ the Nyquist plot cannot encircle -1 and the system must be stable. As ω_{mag} can be made as small as desired by a proper choice of c without affecting ω_{phase}, we find that this controller can always be used to give a system that is closed-loop stable – provided that c is made small enough.

The root-locus analysis

Now consider the root-locus of the system with a phase-lead compensator. Assuming that ω_{o} is small and that $\omega_{\mathrm{p}} > 2/\tau$, we find that, from left to right, there is a zero (of the Padé approximation) at $2/\tau$, two poles (of the automobile model) at 0, a zero (of the compensator) at $-\omega_{\mathrm{o}}$, a pole (of the Padé approximation) at $-2/\tau$ and a pole (of the compensator) at $-\omega_{\mathrm{p}}$. If ω_{o} is very small then we expect its action to be effectively to cancel one of the poles at zero and to make the system behave like the previous system – like a system that uses derivative compensation alone. Of course, this similarity will hold only for relatively small gains. In Fig. 5 the root loci of the system with derivative control and the system with a phase-lead controller are given. (The m-file that generated these figures, `rl_driv.m`, is available at [7].) Note that near the origin the two systems look quite similar.

A Comparison of the derivative and phase-lead controllers

The controllers are:

$$T_1(s) = c_1 s$$

$$T_2(s) = c_2 \frac{s/\omega_{\mathrm{o}} + 1}{s/\omega_{\mathrm{p}} + 1}$$

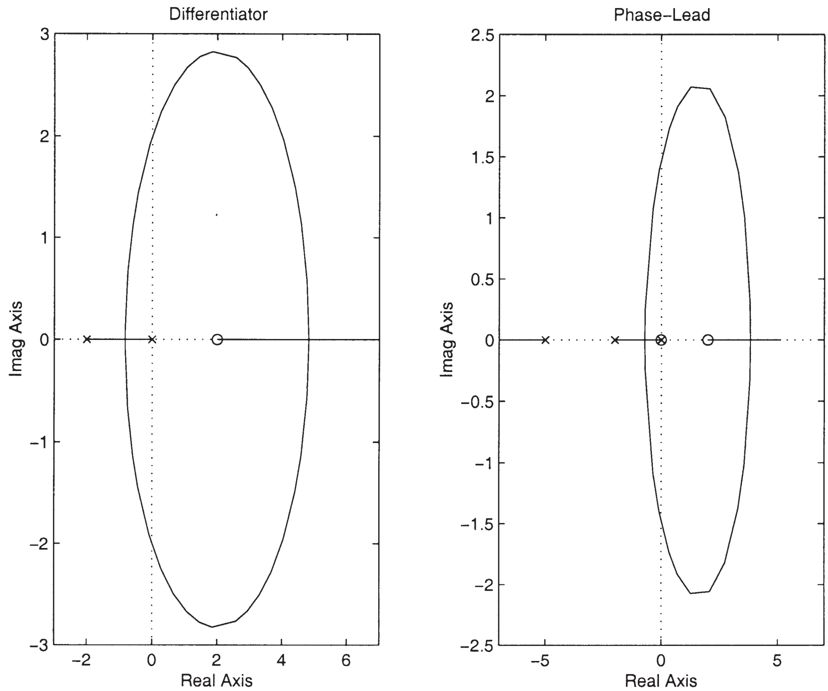

Fig. 5 *A comparison of the root loci. Note the different scales.*

(where the multiplicative constants have been included in the transfer functions). Assuming that ω_o is small (i.e. much smaller than either 1 or ω_p) then as long as s is reasonably small (as long as the frequency is relatively low) it is found that:

$$T_2(s) \approx \frac{c_2}{\omega_o} s$$

This implies that by letting $c_2 = c_1\omega_o$, one should be able to cause the system with the phase-lead controller to behave similarly to the system with pure derivative control. This turns out to be correct.

Next, the stability of the system that is approximately equivalent to the optimal system using derivative control is examined. For that system it was found that:

$$\frac{K_s c_1 v^2 \tau}{l} \approx 0.34$$

In the system with the phase-lead controller let:

$$c_2 = c_1\omega_o = \frac{0.34 l}{K_s v^2}\omega_o$$

Let us consider the stability of this system with $\omega_o = 0.01$, $\omega_p = 5$, and $\tau = 1$. With these choices, it is found that:

$$G_c(s)G_p(s) = 0.0034\frac{100s+1}{s/5+1}e^{-s}\frac{1}{s^2}$$

Using Matlab, the phase and the magnitude of this function were plotted for $s = j\omega$, $\omega \geq 0.015$. The plots are shown in Fig. 6. Note that the phase is always greater than $-180°$ at any location for which the magnitude is greater than 1. Additionally, for the values of the parameters that have been chosen, $\omega_{\text{phase}} = 0.0172$. By definition, for any value of ω that is less than ω_{phase} the phase is greater than $-180°$. Thus, the Nyquist plot of the system cannot encircle -1 and the system is stable.

In Fig. 7 the unit step response of systems for which these choices were made is shown. Note that the output of the two systems is quite similar, though the one with the phase-lead controller has some barely visible oscillations and has not finished settling even after 100 s. (The Simulink models used to generate the step responses, `dr_ser.mdl` for the derivative controller and `driving.mdl` for the phase-lead controller, are available at [7].) The increase in settling time is to be expected as the

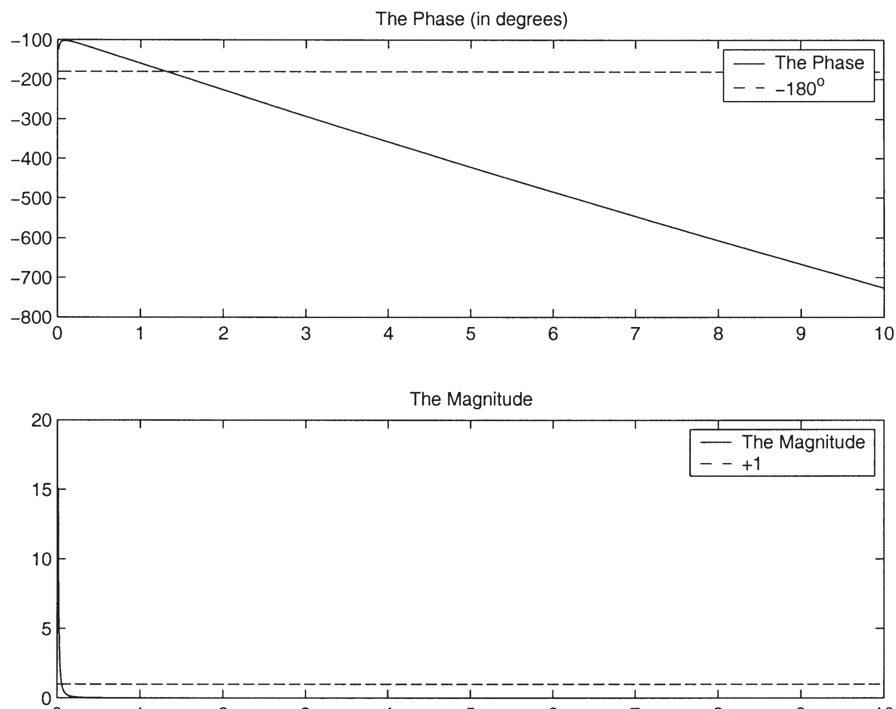

Fig. 6 *The upper figure is a plot of the phase of $G_c(j\omega)G_p(j\omega)$. The lower figure is a plot of the magnitude of $G_c(j\omega)G_p(j\omega)$.*

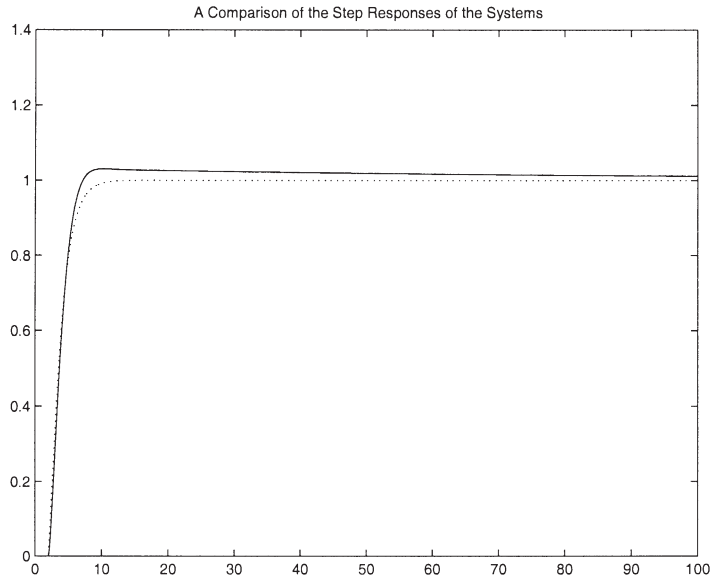

Fig. 7 *A comparison of the step responses of the two systems. The output of the system with derivative control is given as a dotted line.*

phase-lead compensator adds a pole very near zero. (See Fig. 8, in which some points on the root-locus in the neighbourhood of the origin are plotted.) The reason this pole does not cause more trouble is that its coefficient is very small – it is a by-product of the zero of the phase-lead compensator, and the zero is almost cancelled by one of the poles at zero.

A Comparison with a more precise model

In [1] a model for the driver–automobile system is presented. For small values of s the model for the automobile is, as mentioned above, the same as the model presented here. The model of the driver presented in [1] is broken into two parts. The low-frequency part is the PD compensator:

$$1.75(s+0.325)$$

There is also a rather complicated high-frequency part whose gain tends to unity at the low-frequency limit and to 0 at the high-frequency limit.

In the limit of large ω_p, the phase-lead compensator is just a PD compensator. It is found that the models presented here – for both the driver and the automobile – are quite similar to the models given in [1]. The model presented here ignores the high-frequency dynamics of the system except inasmuch as the phase-lead controller has finite high-frequency gain, whereas the PD controller used in [1] has infinite gain at the high-frequency limit. (The PD compensator's high gain at high

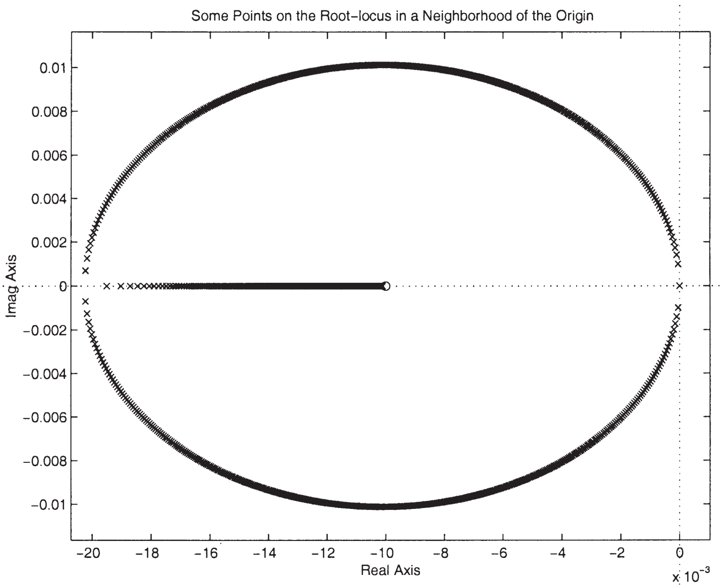

Fig. 8 *Some points in the neighbourhood of the origin that are on the root-locus of the system with a phase-lead compensator.*

frequencies is cancelled by the attenuation provided by the high-frequency portion of their model.)

Classroom notes – the Nyquist plot

Many students have difficult with Nyquist plots. In this presentation, Nyquist plots are used in an essential fashion, and the instructor has the opportunity to review with the students how Nyquist plots are used.

In the section above on the phase-lead controller, the Nyquist plot is used to show that the system with the derivative controller is input–output stable. The discussion is quite complete, but students are often helped by simple diagrams. Plotting the spiraling Nyquist plot that corresponds to the system under consideration helps the students understand why the only point that is really of interest is the point at which the Nyquist plot *first* intersects the negative real axis.

Again, in the section above comparing the derivative and phase-lead controllers, when Fig. 6 is presented, the connection between the figure and the Nyquist diagram may not be clear to the students. A quick sketch of a Nyquist diagram for which the magnitude is alway less than 1 when the phase is less than or equal to -180° makes the reason why the system under consideration is stable clear to the students.

Conclusions

A model of the driver–automobile system is developed, and certain controller models are examined. It is shown that the assumption that the person's transfer function includes a component that behaves like a phase-lead controller is in reasonable agreement with the low-frequency portion of the model developed by Hess and Modjtahedzadeh [1] – a model that was derived from a combination of experiments and a knowledge of physics and physiology.

Such control laws act to try to predict the future – as do drivers. The output of the phase-lead controller is bounded at all frequencies, as is a person's. It is also shown that the reaction time of the driver–automobile system plays an important role in determining how the driver must steer.

This material is the basis for an interesting and informative lecture. The material can be used to show the students the importance of mathematical modelling and the ability of control theory to contribute to our understanding of the world in which we live.

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