Electromagnetic fields theory of electrical machines Part I: Poynting theorem for electromechanical energy conversion

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Abstract The power flow across a closed surface containing moving media is considered using the Poynting theorem. Various components of this power are identified with known expressions. The effects of hysteresis are considered in a linearized form by assuming hypothetical relations between field vectors. These relations result in elliptical hysteresis loops for sinusoidal time-varying fields. The treatment presented may be useful to students preparing for a master's degree or final year bachelor's degree.

Keywords energy conversion; Maxwell's equations; Poynting theorem

The theory of electrical machines found in textbooks is usually based on a lumped circuit approach. Kron and others[1–3] suggested an alternative approach using a tensor–matrix method. A few books[4–6] and many papers[7–11] indicate the possibility of yet a third approach, namely an electromagnetic fields approach to the theory of electrical machines.

The advantages of a fields approach include:

1 It identifies the various approximations involved in the treatment.
2 It supplements the other two approaches by providing more accurate values for circuit parameters.
3 It is an alternative to the existing approach for the treatment of certain special-purpose electrical machines, such as reluctance, hysteresis and eddy current motors.
4 It is amenable to computer based numerical methods.

The objective of this and the companion paper[12] is to provide a unified treatment for the fields approach to the theory of different electrical machines. The essentially nonlinear phenomena of hysteresis[13,14] are considered, albeit approximately, by assuming linear constitutive equations.

The material included in this paper is useful to students preparing for a master's degree or final year bachelor's degree. At this level they learn Poynting's theorem and its application to radiation from antennas. The treatment presented extends the application of Poynting's theorem to electromechanical energy conversion.

Fields in moving media

Electromechanical energy conversion takes place only in moving media. Consider a medium moving with a velocity u, relative to a reference frame. At the instant t, let a point P moving with the medium coincides with a fixed point Q. At this instant, fields are given by

$$E = E' - u \times B' \tag{1.1}$$

and

$$H = H' + u \times D' \tag{1.2}$$

where E', B', H' and D' indicate fields at the moving point measured in the stationary reference frame.

Let the constitutive equations be given as

$$B' = \mu H' + \mu' \hat{H}' \tag{2.1}$$

and

$$D' = \varepsilon E' + \varepsilon' \hat{E}' \tag{2.2}$$

where μ, μ', ε and ε' are positive real scalar functions of space coordinates, while $\hat{H}'$ and $\hat{E}'$ indicate Hilbert transform of H' and E', respectively. For steady-state sinusoidal time-varying electromagnetic fields, these relations result complex permeability and complex permittivity, if field vectors are given in phasor forms.[15]

To a stationary observer, media parameters at the fixed point Q are functions of time as well as the space coordinates of the point Q. Thus

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\partial \mu}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \mu}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \mu}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial \mu}{\partial t} \tag{3}$$

or

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\partial \mu}{\partial t} \tag{3.1}$$

since, for the fixed point

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

To an observer moving with the medium, these parameters are functions only of time-varying space coordinates of the point Q, which is moving with a velocity ($-u$) relative to the medium. Thus

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\partial \mu}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \mu}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \mu}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

or

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = -u\cdot\nabla\mu \tag{3.2}$$

Therefore, from equations (3.1) and (3.2)

$$\frac{\partial\mu}{\partial t} = -u\cdot\nabla\mu \tag{4.1}$$

Similarly

$$\frac{\partial\mu'}{\partial t} = -u\cdot\nabla\mu' \tag{4.2}$$

$$\frac{\partial\varepsilon}{\partial t} = -u\cdot\nabla\varepsilon \tag{4.3}$$

and

$$\frac{\partial\varepsilon'}{\partial t} = -u\cdot\nabla\varepsilon' \tag{4.4}$$

Therefore, in view of equations (2.1), (4.1) and (4.2)

$$\frac{\partial B'}{\partial t} = \mu\frac{\partial H'}{\partial t} + \mu'\frac{\partial \hat{H}'}{\partial t} + (-u\cdot\nabla\mu)H' + (-u\cdot\nabla\mu')\hat{H}' \tag{5.1}$$

and from equations (2.2), (4.3) and (4.4)

$$\frac{\partial D'}{\partial t} = \varepsilon\frac{\partial E'}{\partial t} + \varepsilon'\frac{\partial \hat{E}'}{\partial t} + (-u\cdot\nabla\varepsilon)E' + (-u\cdot\nabla\varepsilon')\hat{E}' \tag{5.2}$$

Further, at the moving point P, since fields are functions of time as well as the coordinates of the moving point P,

$$\frac{\mathrm{d}H'}{\mathrm{d}t} = \frac{\partial H'}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial H'}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial H'}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial H'}{\partial t} \tag{6}$$

therefore

$$\frac{\partial H'}{\partial t} = \frac{\mathrm{d}H'}{\mathrm{d}t} + (-u\cdot\nabla)H' \tag{6.1}$$

Similarly

$$\frac{\partial \hat{H}'}{\partial t} = \frac{\mathrm{d}\hat{H}'}{\mathrm{d}t} + (-u\cdot\nabla)\hat{H}' \tag{6.2}$$

$$\frac{\partial E'}{\partial t} = \frac{\mathrm{d}E'}{\mathrm{d}t} + (-u\cdot\nabla)E' \tag{6.3}$$

and

$$\frac{\partial \hat{E}'}{\partial t} = \frac{\mathrm{d}\hat{E}'}{\mathrm{d}t} + (-u \cdot \nabla)\hat{E}' \tag{6.4}$$

Poynting theorem

The Poynting vector at the instant t, being the same at points P and Q, is obtained as

$$E \times H = (E' - u \times B') \times (H' + u \times D') \tag{7.1}$$

Therefore

$$\nabla \cdot (E \times H) = (H' + u \times D') \cdot [\nabla \times (E' - u \times B')] - (E' - u \times B') \cdot [\nabla \times (H' + u \times D')] \tag{7.2}$$

Using Maxwell's equations for a moving point

$$\nabla \cdot (E \times H) = (H' + u \times D') \cdot \left(-\frac{\partial B'}{\partial t} \right) - (E' - u \times B') \cdot \left(J' + u\rho' + \frac{\partial D'}{\partial t} \right) \tag{8.1}$$

Therefore on simplification

$$-\nabla \cdot (E \times H) = \left(H' \cdot \frac{\partial B'}{\partial t} + E' \cdot \frac{\partial D'}{\partial t} \right) + E' \cdot J' + u \cdot (\rho' E' + J' \times B') + u \cdot \frac{\partial}{\partial t}(D' \times B') \tag{8.2}$$

Now, the term $H' \cdot \dfrac{\partial B'}{\partial t}$, on the RHS of this equation can be written as

$$H' \cdot \frac{\partial B'}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2} H' \cdot B' \right) + \frac{1}{2}\left(H' \cdot \frac{\partial B'}{\partial t} - B' \cdot \frac{\partial H'}{\partial t} \right) \tag{9}$$

Thus, in view of equations (2.1), (4.1), (4.2) and (5.1)

$$H' \cdot \frac{\partial B'}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2} H' \cdot B' \right) + \frac{1}{2}\mu'\left(H' \cdot \frac{\partial \hat{H}'}{\partial t} - \hat{H}' \cdot \frac{\partial H'}{\partial t} \right) + \frac{1}{2}(-u \cdot \nabla\mu)H'^2 + \frac{1}{2}(-u \cdot \nabla\mu')(H' \cdot \hat{H}') \tag{9.1}$$

Similarly, using equations (2.2), (4.3), (4.4) and (5.2)

$$E' \cdot \frac{\partial D'}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2} E' \cdot D' \right) + \frac{1}{2}\varepsilon'\left(E' \cdot \frac{\partial \hat{E}'}{\partial t} - \hat{E}' \cdot \frac{\partial E'}{\partial t} \right) + \frac{1}{2}(-u \cdot \nabla\varepsilon)E'^2 + \frac{1}{2}(-u \cdot \nabla\varepsilon')(E' \cdot \hat{E}') \tag{9.2}$$

where

$$H'^2 = H' \cdot H' \tag{9.3}$$

and

$$E'^2 = E' \cdot E' \tag{9.4}$$

Using equations (6.1), (6.2) and (9.1)

$$\begin{aligned} H' \cdot \frac{\partial B'}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2}H' \cdot B'\right) + \frac{1}{2}\mu'\left(H' \cdot \frac{\mathrm{d}\hat{H}'}{\mathrm{d}t} - \hat{H}' \cdot \frac{\mathrm{d}H'}{\mathrm{d}t}\right) \\ &\quad + \frac{1}{2}\mu'\left[H' \cdot (-u \cdot \nabla)\hat{H}' - \hat{H}' \cdot (-u \cdot \nabla)H'\right] \\ &\quad + \frac{1}{2}(-u \cdot \nabla\mu)H'^2 + \frac{1}{2}(-u \cdot \nabla\mu')\left(H' \cdot \hat{H}'\right) \end{aligned} \tag{10}$$

or

$$\begin{aligned} H' \cdot \frac{\partial B'}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2}H' \cdot B'\right) + \frac{1}{2}\mu'\left(H' \cdot \frac{\mathrm{d}\hat{H}'}{\mathrm{d}t} - \hat{H}' \cdot \frac{\mathrm{d}H'}{\mathrm{d}t}\right) \\ &\quad + u \cdot \left[\frac{1}{2}\mu' a_u\left\{H' \cdot (-a_u \cdot \nabla)\hat{H}' - \hat{H}' \cdot (-a_u \cdot \nabla)H'\right\}\right] \\ &\quad + u \cdot \left[\frac{1}{2}(-\nabla\mu)H'^2 + \frac{1}{2}(-\nabla\mu')\left(H' \cdot \hat{H}'\right)\right] \end{aligned} \tag{10.1}$$

Similarly

$$\begin{aligned} E' \cdot \frac{\partial D'}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{2}E' \cdot D'\right) + \frac{1}{2}\varepsilon'\left(E' \cdot \frac{\mathrm{d}\hat{E}'}{\mathrm{d}t} - \hat{E}' \cdot \frac{\mathrm{d}E'}{\mathrm{d}t}\right) \\ &\quad + u \cdot \left[\frac{1}{2}\varepsilon' a_u\left\{E' \cdot (-a_u \cdot \nabla)\hat{E}' - \hat{E}' \cdot (-a_u \cdot \nabla)E'\right\}\right] \\ &\quad + u \cdot \left[\frac{1}{2}(-\nabla\varepsilon)E'^2 + \frac{1}{2}(-\nabla\varepsilon')\left(E' \cdot \hat{E}'\right)\right] \end{aligned} \tag{10.2}$$

where a_u indicates the unit vector at P, parallel to the velocity vector u.

On integrating both sides of equation (8.2) over a source-free region of volume v bounded by its surface S, and then using the divergence theorem one obtains, using equations (10.1) and (10.2)

$$-\int_S (E \times H) \cdot \mathrm{d}s = \sum_{n=1}^{10} \int_r \wp_n \,\mathrm{d}v \tag{11}$$

where

$$\wp_1 = \frac{\partial}{\partial t}\left(\frac{1}{2}H' \cdot B' + \frac{1}{2}E' \cdot D'\right) \tag{11.1}$$

$$\wp_2 = E' \cdot J' \tag{11.2}$$

$$\wp_3 = \frac{1}{2}\mu'\left(H' \cdot \frac{\mathrm{d}\hat{H}'}{\mathrm{d}t} - \hat{H}' \cdot \frac{\mathrm{d}H'}{\mathrm{d}t}\right) \tag{11.3}$$

$$\wp_4 = \frac{1}{2}\varepsilon'\left(E'\cdot\frac{\mathrm{d}\hat{E}'}{\mathrm{d}t} - \hat{E}'\cdot\frac{\mathrm{d}E'}{\mathrm{d}t}\right) \tag{11.4}$$

$$\wp_5 = u\cdot\frac{\partial}{\partial t}(D'\times B') \tag{11.5}$$

$$\wp_6 = u\cdot[\rho'E' + J'\times B'] \tag{11.6}$$

$$\wp_7 = u\cdot\left[\frac{1}{2}\mu' a_u\left\{H'\cdot(-a_u\cdot\nabla)\hat{H}' - \hat{H}'\cdot(-a_u\cdot\nabla)H'\right\}\right] \tag{11.7}$$

$$\wp_8 = u\cdot\left[\frac{1}{2}\varepsilon' a_u\left\{E'\cdot(-a_u\cdot\nabla)\hat{E}' - \hat{E}'\cdot(-a_u\cdot\nabla)E'\right\}\right] \tag{11.8}$$

$$\wp_9 = u\cdot\left[\frac{1}{2}(-\nabla\mu)H'^2 + \frac{1}{2}(-\nabla\mu')\left(H'\cdot\hat{H}'\right)\right] \tag{11.9}$$

and

$$\wp_{10} = u\cdot\left[\frac{1}{2}(-\nabla\varepsilon)E'^2 + \frac{1}{2}(-\nabla\varepsilon')\left(E'\cdot\hat{E}'\right)\right] \tag{11.10}$$

The LHS of equation (11) indicates the total power entering into volume v through its surface S. The RHS of this equation gives the components of this power.

Components of power transfer

The term $\wp_n$ in equation (11) indicates power density in the volume v. There are ten terms on the RHS of this equation. Each corresponds to a different component of the power. Components of this power can be identified with the help of equations (11.1)–(11.10).

The first term on the RHS gives the rate of increase of stored electromagnetic energy inside the volume.

The second term gives ohmic losses within the volume, while the third and fourth terms give hysteresis losses due to time-varying magnetic and electric fields, respectively.

The fifth term indicates transfer of electromagnetic momentum into the volume v across the surface S.

The first part of the sixth term gives mechanical power used to move charged bodies in the presence of the electric field, while the second part gives the mechanical power expended to move current-carrying conductors in the presence of the magnetic field.

The seventh and eighth terms give the mechanical power developed due to the effects of magnetic and dielectric hysteresis, respectively.

The remaining two terms give the mechanical power developed due to space variation in magnetic and dielectric parameters of the medium.

Electromagnetic forces

Considering coefficients of the velocity vector u in equation (11), force per unit volume $\Im$, is given by

$$\Im = \left[\frac{\partial}{\partial t}(D' \times B')\right] + \left[\rho' E' + J' \times B'\right] + \Im_{mh} + \left[\frac{1}{2}(-\nabla\mu)H'^2 + \frac{1}{2}(-\nabla\mu')\left(H' \cdot \hat{H}'\right)\right]$$
$$+ \Im_{eh} + \left[\frac{1}{2}(-\nabla\varepsilon)E'^2 + \frac{1}{2}(-\nabla\varepsilon')\left(E' \cdot \hat{E}'\right)\right] \qquad (12)$$

where, expanded forms for $\Im_{mh}$ and $\Im_{eh}$ are

$$\Im_{mh} = a_x \frac{1}{2}\mu'\left\{\hat{H}' \cdot \frac{\partial}{\partial x} H' - H' \cdot \frac{\partial}{\partial x}\hat{H}'\right\} + a_y \frac{1}{2}\mu'\left\{\hat{H}' \cdot \frac{\partial}{\partial y} H' - H' \cdot \frac{\partial}{\partial y}\hat{H}'\right\}$$
$$+ a_z \frac{1}{2}\mu'\left\{\hat{H}' \cdot \frac{\partial}{\partial z} H' - H' \cdot \frac{\partial}{\partial z}\hat{H}'\right\} \qquad (12.1)$$

and

$$\Im_{eh} = a_x \frac{1}{2}\varepsilon'\left\{\hat{E}' \cdot \frac{\partial}{\partial x} E' - E' \cdot \frac{\partial}{\partial x}\hat{E}'\right\} + a_y \frac{1}{2}\varepsilon'\left\{\hat{E}' \cdot \frac{\partial}{\partial y} E' - E' \cdot \frac{\partial}{\partial y}\hat{E}'\right\}$$
$$+ a_z \frac{1}{2}\varepsilon'\left\{\hat{E}' \cdot \frac{\partial}{\partial z} E' - E' \cdot \frac{\partial}{\partial z}\hat{E}'\right\} \qquad (12.2)$$

a_x, a_y and a_z indicate unit vectors in the x, y and z directions.

The first term on the RHS of equation (12) indicates the rate of increase of electromagnetic momentum per unit volume. The second term is the Lorentz's force. The second part of this term, $J' \times B'$, governs the principle of operation of most general-purpose electrical machines, such as DC machines, induction and synchronous machines. The third and fourth terms give the force developed due to magnetic hysteresis and that due to space variation of the magnetic parameters of the medium. Likewise, the fifth and sixth terms give the force developed due to electric hysteresis and that due to space variation of the dielectric parameters of the medium. The operation of hysteresis motors is based on the third term, while the fourth term governs the principle of operation of reluctance motors. The last two terms indicate development of forces in dielectric motors.

Concluding remarks

The Poynting theorem is used extensively to determine power radiated from antenna or eddy current losses in a conducting medium. In the present treatment the Poynting theorem is extended to slowly moving media. As a result, non-relativistic expressions for electromagnetic forces are obtained. Most of these expressions found in textbooks[16,17] are derived by different methods.

To include an approximate treatment for certain effects of hysteresis, hypothetical relations between B and H, and between D and E vectors are assumed. These relations result in elliptical hysteresis loops for sinusoidal time-varying fields. It may, however, be seen that the theorem is equally valid for electromagnetic fields in hysteresis-free media.

The expression for the force density $\Im$ satisfies equation (11) even if an arbitrary vector, which is normal to the velocity vector u, is added to it. It is thus tacitly assumed that electromagnetic fields cannot produce an arbitrary force that does not take part in electromechanical energy conversion.

In the circuit approach to the theory of induction machines, both the rotor copper loss as well as the mechanical power developed are treated in the equivalent circuit as power dissipated in resistors. The field approach, however, treats the two power components differently. The former can be found from $\wp_2$ and the latter from $\wp_6$.

The treatment presented is for general time-varying fields in a finite, inhomogeneous moving medium. However, for the interpretation of various terms in equation (11) use is made of energy density expressions, which are obtained for time-invariant fields in homogeneous stationary medium extending over infinite volume. This usage seems to be justified since expressions for the various components of power thus found, correspond to those derived in textbooks by different methods.

Lastly, it may be seen that equation (11) remains unaltered if curl of an arbitrary vector is added to the Poynting vector $E \times H$. Patently, no unique meaning can be attached to the Poynting vector as such. The Poynting theorem simply states that integration of the vector $(-E \times H)$ over a closed surface, bounding a source-free region, gives inflow of power with components indicated on the RHS of equation (11).

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