Teaching to undergraduates the optimum power transfer to a load under constraints

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Abstract This paper deals with the optimum power transfer to a load under constraints using a mathematical simulation model. The algorithm is given in flowchart form to aid the educational process. A numerical example is also provided.

Keywords load matching algorithm; load matching under constraints; optimum power transfer

Based on Thevenin's theorem and under the assumption that all sources operate at the same frequency, any linear circuit at sinusoidal steady state (see Fig. 1) is equivalent to the circuit shown in Fig. 2.[1–10]

The load impedance may represent an electric motor, an antenna, a TV or the input impedance of an electric appliance.

The impedances Z_{TH} and Z_L appearing in Fig. 2 are given by

$$Z_{TH} = R_{TH} + jX_{TH} \tag{1}$$

$$Z_L = R_L + jX_L \tag{2}$$

There are applications in areas such as communications where it is desirable to maximize the power delivered to a load. Assuming that the load impedance $Z_L(R_L, X_L)$ is variable, maximum power transfer is achieved by maximizing the function:

$$P(R_L, X_L) = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}\frac{|\mathbf{V}_{TH}|^2}{(R_{TH}+R_L)^2 + (X_{TH}+X_L)^2} R_L \tag{3}$$

Maximization of $P(R_L, X_L)$ is achieved by solving the system of equations

$$\frac{\partial P(R_L, X_L)}{\partial R_L} = 0 \quad \text{and} \quad \frac{\partial P(R_L, X_L)}{\partial X_L} = 0$$

It is easy to find that the maximum occurs when Z_L is equal to the conjugate of Z_{TH},[1–9,11–13] that is when

$$Z_L = Z^*_{TH} = R_{TH} - jX_{TH} \tag{4}$$

The maximum average power transferred to the load is found to be:

$$P_{max} = P(R_{TH}, -X_{TH}) = \frac{|\mathbf{V}_{TH}|^2}{8R_{TH}} \tag{5}$$

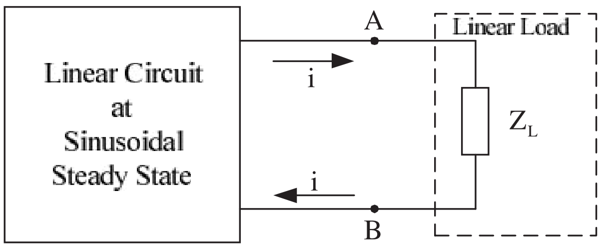

Fig. 1 *Linear circuit at sinusoidal steady state.*

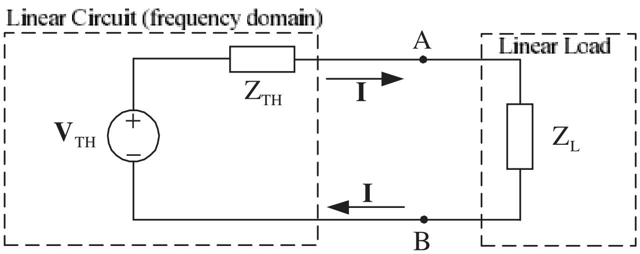

Fig. 2 *Thevenin equivalent for the circuit of Fig. 1.*

It is also well known that load matching can be achieved using a pure resistive load R_L. In this case the maximum occurs when[1–3]

$$R_L = |Z_{TH}| = \sqrt{R_{TH}^2 + X_{TH}^2} \tag{6}$$

When the imaginary part X_L of Z_L is constant and the real part R_L is variable, matching can also be achieved when[3]

$$R_L = |Z_{TH} + jX_L| = \sqrt{R_{TH}^2 + (X_{TH} + X_L)^2} \tag{7}$$

When R_L and X_L are subject to the following constraints:

$$\begin{aligned} R_1 \le R_L \le R_2 \\ X_1 \le X_L \le X_2 \end{aligned} \tag{8}$$

where

- $R_1 \ge 0,\ R_2 > 0$

and

- $X_1 \cdot X_2 \ge 0$ or $X_1 < 0,\ X_2 \le 0$ if $X_{TH} > 0$ (capacitive matching load)
  $X_1 \ge 0,\ X_2 > 0$ if $X_{TH} < 0$ (inductive matching load)

maximization of $P(R_L, X_L)$ given by equation (3) must be achieved under these constraints.

In this paper, the optimum power transfer under constraints, occurring in many real-life engineering applications, is examined by employing a simple mathematical model. Depending on the position of the unconditional maximum (R_{TH}, $-X_{TH}$) with respect to constraints given by (8), several cases result and are dealt with. Only some of these cases are dealt with in a comprehensive way in the literature; some relevant rules of thumb are encountered in certain electric circuit textbooks.[1,3]

Mathematical formulation of the physical problem

Taking into account equations (3) and (8), mathematical modelling of the physical problem is:

Given the function

$$f(x, y) = \frac{kx}{(a+x)^2 + (b+y)^2} \tag{9}$$

where:

$$k, a \in R^+ \qquad b \in R$$

$$x_1 \le x \le x_2 \text{ with } x_1, x_2 \in R^+ \qquad y_1 \le y \le y_2 \text{ with } y_1, y_2 \in R \text{ and } y_1 \cdot y_2 \ge 0$$

Determine the values of x and y maximizing $f(x, y)$ and the value of the maximum.

Solution

When there are no constraints, that is when $f(x, y)$ is defined in $[0, \infty) \times (-\infty, 0]$ or $[0, \infty) \times [0, \infty)$, the maximum resulting from the solution of the simultaneous equations $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ occurs when $(x, y) = (a, -b)$. The maximum value is

$$f_{\max} = f(a, -b) = \frac{k}{4a} \tag{10}$$

However, if there are constraints, the constrained maximum depends on the location of point $(a, -b)$ with respect to region $[x_1, x_2] \times [y_1, y_2]$ as shown in Fig. 3.[14,15] All possible regions (nine) where point $(a, -b)$ can be located are shown in Fig. 3.

When the point $(a, -b) \notin [x_1, x_2] \times [y_1, y_2]$ (region 1, Fig. 3), and given that this region is compact, the maxima or minima of $f(x, y)$ will appear at its boundary.[16] Indeed, if the function was to have a maximum or minimum in the interior of region 1, then the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ would be zero at an interior point of this region. However, this is not the case since these derivatives are zero at point $(a, -b)$, which does not belong to this particular region.

Function $f(x, y)$ at the boundary of region 1 is of the form:

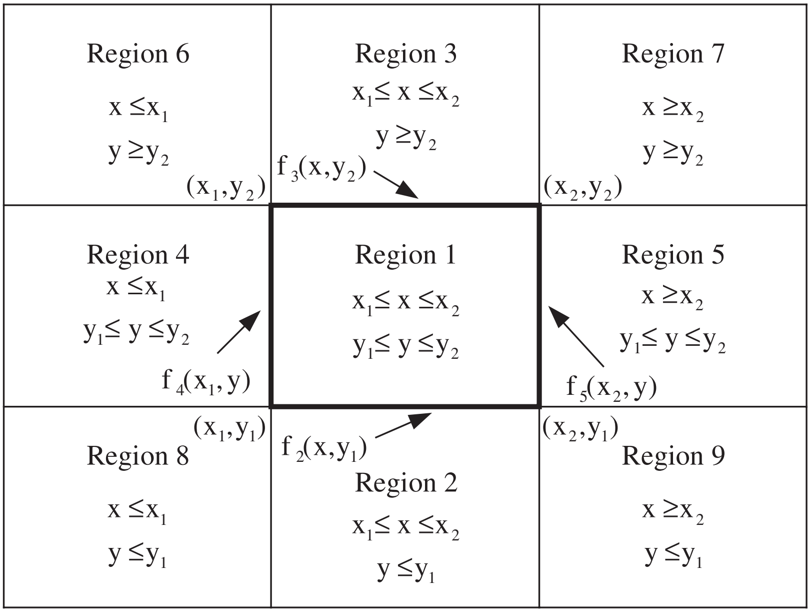

Fig. 3 *Regions where point (a, −b) is probably to be found.*

$$f(x,y)=\begin{cases} f_2(x,y_1)=\dfrac{kx}{(a+x)^2+(b+y_1)^2}, & \text{defined at } [x_1,x_2]\times\{y_1\} \\ f_3(x,y_2)=\dfrac{kx}{(a+x)^2+(b+y_2)^2}, & \text{defined at } [x_1,x_2]\times\{y_2\} \\ f_4(x_1,y)=\dfrac{kx_1}{(a+x_1)^2+(b+y)^2}, & \text{defined at } \{x_1\}\times[y_1,y_2] \\ f_5(x_2,y)=\dfrac{kx_2}{(a+x_2)^2+(b+y)^2}, & \text{defined at } \{x_2\}\times[y_1,y_2] \end{cases} \quad (11)$$

The following cases are then determined.

When $(a, -b)$ is located in region 1, the maximum is determined by maximizing the function $f(x, y)$. The location of maximum is at point $(a, -b)$ and the maximum value is given by equation (10).

When $(a, -b)$ is located in regions 2, 3, 4 or 5, the constrained maximum is determined by maximizing the functions $f_2(x, y_1)$, $f_3(x, y_2)$, $f_4(x_1, y)$, and $f_5(x_2, y)$, respectively. The location of maximum and the maximum value for each of the above cases are shown in Table 1.

When $(a, -b)$ is located in regions 6, 7, 8 or 9, the point maximizing $f(x, y)$ under the corresponding constraint is the only common point of the relevant region and region 1. This point is the corresponding corner point of region 1. With this remark taken into account, the location of maximum and the maximum value for each of the above cases are easily formed and are shown in Table 1.

TABLE 1 *The location of maximum and the maximum value*

Location of $(a, -b)$	Function	Location of maximum	Maximum value
Region 1	$f(x, y)$	$(a, -b)$	$f(a,-b)=\frac{k}{4a}$
Region 2	$f_2(x, y_1)$	$(d_2, y_1)=\left(\sqrt{a^2+(b+y_1)^2},\ y_1\right)$	$f_2(d_2, y_1)=\frac{k}{2(a+d_2)}$
Region 3	$f_3(x, y_2)$	$(d_3, y_2)=\left(\sqrt{a^2+(b+y_2)^2},\ y_2\right)$	$f_3(d_3, y_2)=\frac{k}{2(a+d_3)}$
Region 4	$f_4(x_1, y)$	$(x_1, -b)$	$f_4(x_1,-b)=\frac{kx_1}{(a+x_1)^2}$
Region 5	$f_5(x_2, y)$	$(x_2, -b)$	$f_5(x_2,-b)=\frac{kx_2}{(a+x_2)^2}$
Region 6	$f(x, y)$	(x_1, y_2)	$f(x_1, y_2)=\frac{kx_1}{(a+x_1)^2+(b+y_2)^2}$
Region 7	$f(x, y)$	(x_2, y_2)	$f(x_2, y_2)=\frac{kx_2}{(a+x_2)^2+(b+y_2)^2}$
Region 8	$f(x, y)$	(x_1, y_1)	$f(x_1, y_1)=\frac{kx_1}{(a+x_1)^2+(b+y_1)^2}$
Region 9	$f(x, y)$	(x_2, y_1)	$f(x_2, y_1)=\frac{kx_2}{(a+x_2)^2+(b+y_1)^2}$

Physical problem (optimum matching)

Based on the previous mathematical formulation of the physical problem and making the substitutions

$$
\begin{aligned}
&a \to R_{\mathrm{TH}} \quad b \to X_{\mathrm{TH}} \quad k \to \frac{1}{2}|\mathbf{V}_{\mathrm{TH}}|^2 \\
&x \to R_{\mathrm{L}} \quad y \to X_{\mathrm{L}} \quad f(x, y) \to P(R_{\mathrm{L}}, X_{\mathrm{L}})
\end{aligned}
$$

the problem of optimum power transfer to a load under the constraints given by equation (8), has nine possible solutions depending on the location of $(R_{\mathrm{TH}}, -X_{\mathrm{TH}})$ with respect to the region where the variables R_{L} and X_{L} are constrained.

However, teaching this subject requires an easily understandable presentation of the solution. Beyond the mathematical analysis one will finally be interested in applying the solution to real cases. For this purpose, Fig. 4 presents an algorithm in flowchart form resulting from the mathematical analysis and leading to all possible solutions for optimum load matching. Through this algorithm, students or engineers can easily obtain the optimum matching for a particular application.

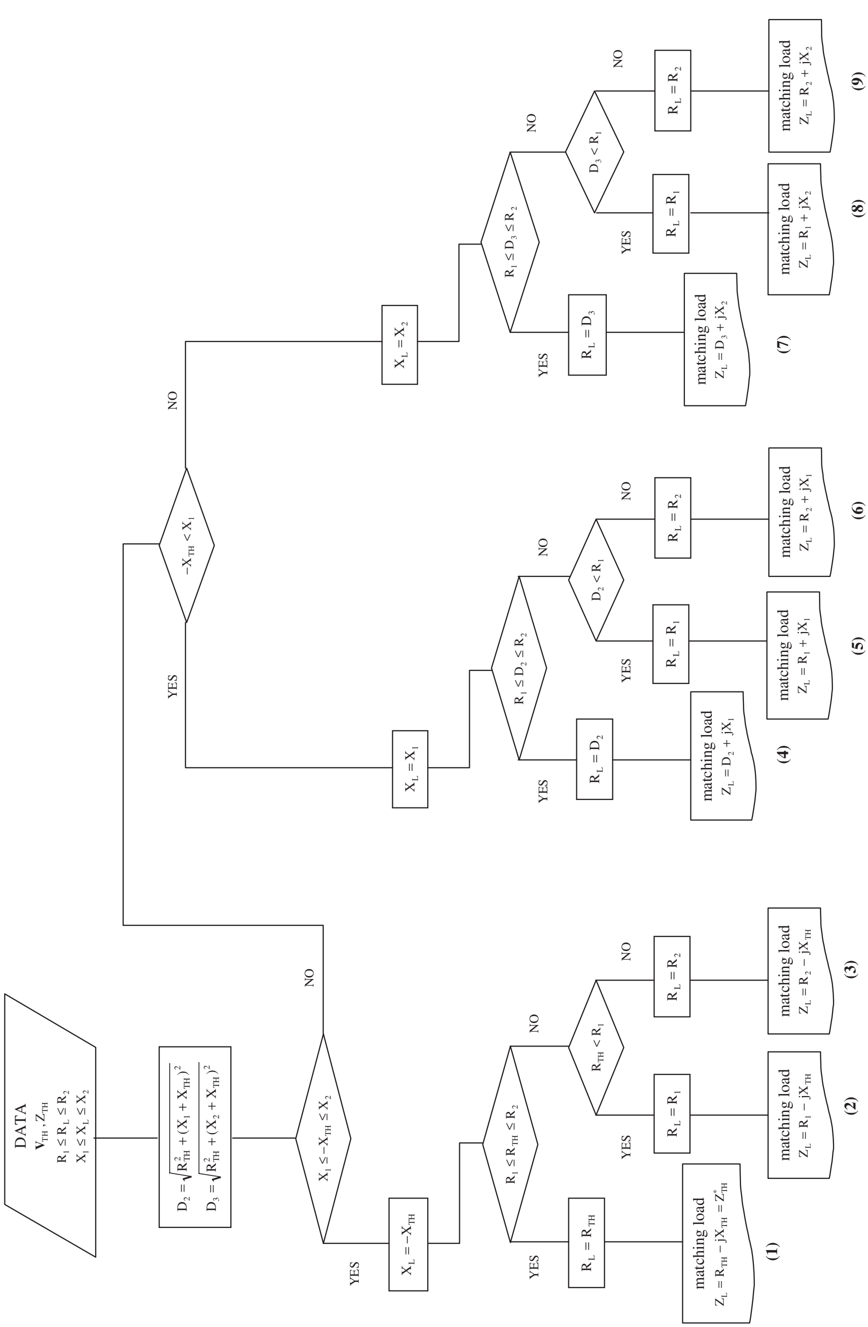

Fig. 4 *Flowchart for finding the optimum matching load.*

Application of the method

For an electric circuit at sinusoidal steady state supplying a linear load $Z_L = R_L + jX_L$, the Thevenin equivalent circuit is as shown in Fig. 5, where $|\mathbf{V}_{TH}| = 20\,V$.

Calculate the matching load for optimum power transfer and the optimum power transferred to this load with the following constraints:

(1) $0 \le R_L \le 25\,\Omega$, $-50 \le X_L \le 0\,\Omega$ **(2)** $0 \le R_L \le 12.5\,\Omega$, $-25 \le X_L \le 0\,\Omega$ **(3)** $0 \le R_L \le 25\,\Omega$, $-85 \le X_L \le -35\,\Omega$

(4) $10 \le R_L \le 35\,\Omega$, $-50 \le X_L \le 0\,\Omega$ **(5)** $0 \le R_L \le 4.5\,\Omega$, $-45 \le X_L \le -20\,\Omega$ **(6)** $10 \le R_L \le 35\,\Omega$, $-85 \le X_L \le -35\,\Omega$

(7) $0 \le R_L \le 4.5\,\Omega$, $-60 \le X_L \le -35\,\Omega$ **(8)** $10 \le R_L \le 35\,\Omega$, $-25 \le X_L \le 0\,\Omega$ **(9)** $0 \le R_L \le 4.5\,\Omega$, $-25 \le X_L \le 0\,\Omega$

Solution

Following the algorithm presented, the matching load for optimum power transfer and the optimum power transferred to the load for each of the above cases are shown in Figs 6–14:

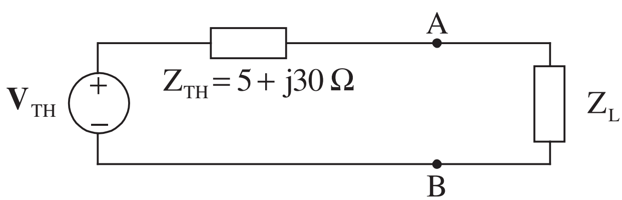

Fig. 5 *Application (Thevenin equivalent circuit).*

$(5,-30) \in [0,25] \times [-50,0]$
(region 1 or output 1 of the flowchart)

$$Z_L = Z_{TH}^* = 5 - j30\,\Omega$$

$$P_{max} = \frac{|\mathbf{V}_{TH}|^2/2}{4R_{TH}} = \frac{200}{4 \cdot 5} = 10\,W$$

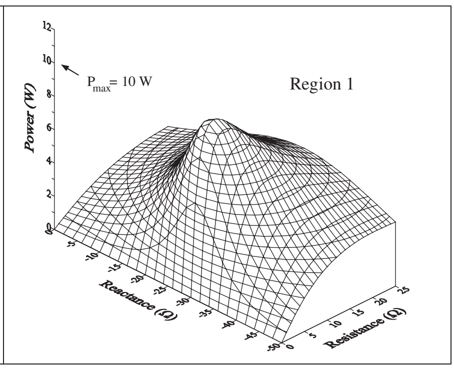

Fig. 6 *Optimum power transfer for case 1.*

$$(5,-30) \in [0,12.5] \times \{-30 < -25\}$$
(region 2 or output 4 of the flowchart)

$$Z_L = \sqrt{R_{TH}^2 + (X_{TH} + X_1)^2} + jX_1 = $$
$$= 7.071 - j25\,\Omega$$

$$P_{max} = \frac{|\mathbf{V}_{TH}|^2 / 2}{2(R_{TH} + \sqrt{R_{TH}^2 + (X_{TH} + X_1)^2})} =$$
$$= 8.284\text{ W}$$

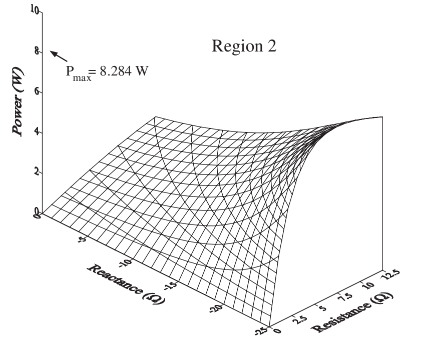

Fig. 7 *Optimum power transfer for case 2.*

$$(5,-30) \in [0,25] \times \{-30 > -35\}$$
(region 3 or output 7 of the flowchart)

$$Z_L = \sqrt{R_{TH}^2 + (X_{TH} + X_2)^2} + jX_2 = $$
$$= 7.071 - j35\,\Omega$$

$$P_{max} = \frac{|\mathbf{V}_{TH}|^2 / 2}{2(R_{TH} + \sqrt{R_{TH}^2 + (X_{TH} + X_2)^2})} =$$
$$= 8.284\text{ W}$$

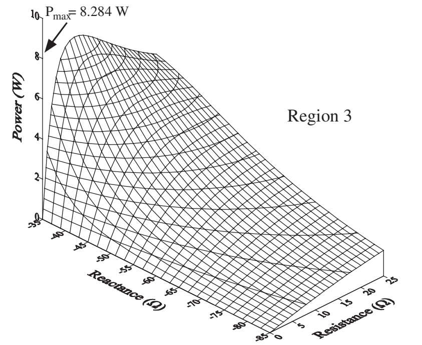

Fig. 8 *Optimum power transfer for case 3.*

$$(5,-30) \in \{5 < 10\} \times [-50,0]$$
(region 4 or output 2 of the flowchart)

$$Z_L = R_1 - jX_{TH} = 10 - j30\,\Omega$$

$$P_{max} = \frac{|\mathbf{V}_{TH}|^2 / 2 \cdot R_1}{(R_{TH} + R_1)^2} = 8.888\text{ W}$$

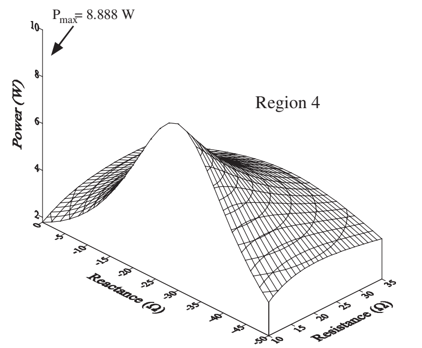

Fig. 9 *Optimum power transfer for case 4.*

$$(5,-30) \in \{5 > 4.5\} \times [-45,-20]$$
(region 5 or output 3 of the flowchart)

$$Z_L = R_2 - jX_{TH} = 4.5 - j30\,\Omega$$

$$P_{max} = \frac{|\mathbf{V}_{TH}|^2 / 2 \cdot R_2}{(R_{TH} + R_2)^2} = 9.972\text{ W}$$

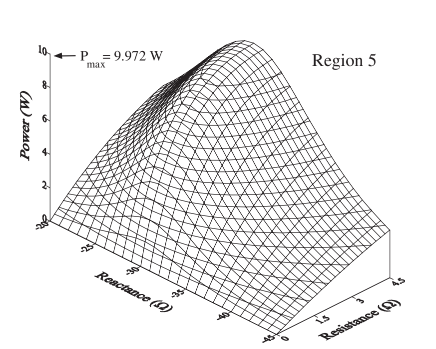

Fig. 10 *Optimum power transfer for case 5.*

$$(5,-30) \in \{5 < 10\} \times \{-30 > -35\}$$
(region 6 or output 8 of the flowchart)

$$Z_L = R_1 + jX_2 = 10 - j35\,\Omega$$

$$P_{max} = \frac{|\mathbf{V}_{TH}|^2 / 2 \cdot R_1}{(R_{TH} + R_1)^2 + (X_{TH} + X_2)^2} = 8\text{ W}$$

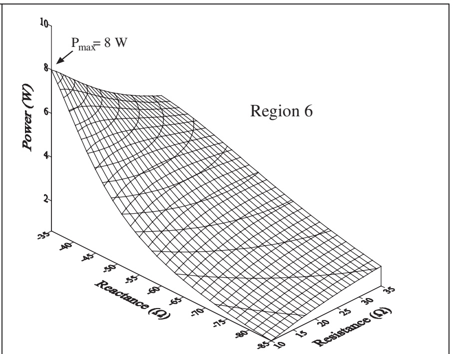

Fig. 11 *Optimum power transfer for case 6.*

$$(5,-30) \in \{5 > 4.5\} \times \{-30 > -35\}$$
(region 7 or output 9 of the flowchart)

$$Z_L = R_2 + jX_2 = 4.5 - j35\,\Omega$$

$$P_{max} = \frac{|\mathbf{V}_{TH}|^2 / 2 \cdot R_2}{(R_{TH} + R_2)^2 + (X_{TH} + X_2)^2} = $$
$$= 7.809\text{ W}$$

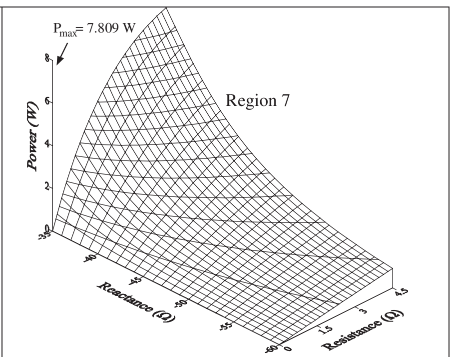

Fig. 12 *Optimum power transfer for case 7.*

$$(5,-30) \in \{5 < 10\} \times \{-30 < -25\}$$
(region 8 or output 5 of the flowchart)

$$Z_L = R_1 + jX_1 = 10 - j25\,\Omega$$

$$P_{max} = \frac{|\mathbf{V}_{TH}|^2 / 2 \cdot R_1}{(R_{TH} + R_1)^2 + (X_{TH} + X_1)^2} = 8\ \text{W}$$

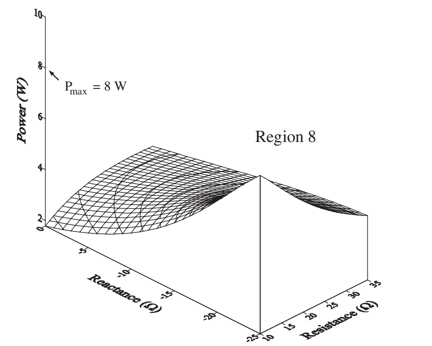

Fig. 13 *Optimum power transfer for case 8.*

$$(5,-30) \in \{5 > 4.5\} \times \{-30 < -25\}$$
(region 9 or output 6 of the flowchart)

$$Z_L = R_2 + jX_1 = 4.5 - j25\,\Omega$$

$$P_{max} = \frac{|\mathbf{V}_{TH}|^2 / 2 \cdot R_2}{(R_{TH} + R_2)^2 + (X_{TH} + X_1)^2} = \\ = 7.809\ \text{W}$$

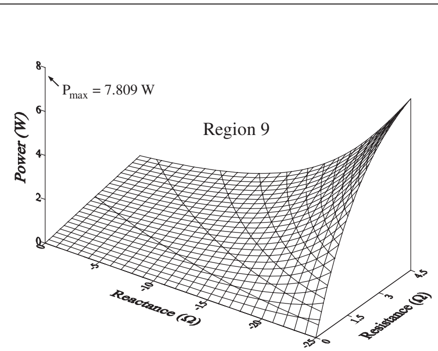

Fig. 14 *Optimum power transfer for case 9.*

Conclusions

From the previous analysis the following conclusions are derived concerning optimum power transfer to a load under constraints:

1 If $R_{TH} \notin [R_1, R_2]$ and $-X_{TH} \notin [X_1, X_2]$, the constrained matching load is represented by one of the four corner points of the orthogonal region $[R_1, R_2] \times [X_1, X_2]$. The appropriate corner point in each case is the only common point of region 1 and the region containing the point $(R_{TH}, -X_{TH})$ (outputs 5, 6, 8, 9 of the flowchart or regions 8, 9, 6, 7, respectively).

2 If $R_{TH} \notin [R_1, R_2]$ and $-X_{TH} \in [X_1, X_2]$, the constrained matching load is always $X_L = -X_{TH}$ and the value of R_L is the nearest to the interval $[R_1, R_2]$ value of R_{TH} (outputs 2, 3 of the flowchart or regions 4, 5, respectively).

3 If $R_{TH} \in [R_1, R_2]$ and $-X_{TH} \notin [X_1, X_2]$, the constrained matching load has as X_L the nearest to the interval $[X_1, X_2]$ value of $-X_{TH}$ and $R_L = \sqrt{R_{TH}^2 + (X_{TH} + X_L)^2}$ (outputs 4,7 of the flowchart or regions 2, 3, respectively). These cases corre-

spond to the matching to a load Z_L for which $\mathrm{Im}(Z_L) = X_L$ = constant and its real part R_L is variable, as discussed in the introduction (equation (7)).

4 If $R_{TH} \in [R_1, R_2]$ and $-X_{TH} \in [X_1, X_2]$, the matching load is $Z_{TH} = Z^*_{TH}$ (output 1 of the flowchart or region 1, respectively). This case is equivalent to the unconditional matching, as referred to in the introduction (equations (4), (5)).

5 If $X_1 = X_2 = 0$ and $|Z_{TH}| \in [R_1, R_2]$, the matching load is $R_L = |Z_{TH}| = \sqrt{R_{TH}^2 + X_{TH}^2}$. This case is equivalent to the matching case for a purely resistive load Z_L, as referred to in the introduction (equation (6)).

6 Finally, all of the above conclusions are subjected to the following rule of thumb: during the process of finding the most suitable matching load, engineers seek to minimize the term $(X_{TH} + X_L)^2$ of equation (3) by selecting appropriately X_L. Next, the value of R_L is chosen depending on the region at which point $(R_{TH}, -X_{TH})$ is located.

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