How to simplify differentiation when performing partial fraction expansion in the presence of a single multiple pole

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Abstract A new method is suggested to handle partial fraction expansion in the presence of a single high order pole, by using polynomial division and partial fraction expansion with simple poles only, to simplify the process of repeated differentiation of a rational function.

Keywords differentiation; partial fraction expansion

Partial fraction expansion of proper rational functions is a powerful and long-established technique for inversion of Laplace transforms in the solution of problems in dynamics of linear systems of many sorts, such as electric circuits and control systems. It is part of the stock-in-trade of every undergraduate in electrical engineering. When simple poles only are involved, the residues, which completely specify the expansion, are easily obtained using Heaviside's famous 'cover-up' rule. In the presence of a multiple pole, generalisation of the expansion proceeds in the following way, which has been presented in the vast majority of textbooks spanning the years from the author's time as an undergraduate to the present.[1,2]

Let

$$Y(s) = KN(s)/D(s) \tag{1}$$

be a ***proper*** rational function in the complex variable s, where $N(s)$ of degree m and $D(s)$ of degree n are monic polynomials, with $m < n$, and K is a scalar, and let $D(s)$ have the factorisation

$$D(s) = (s - p_1)(s - p_2)\cdots(s - p_{k-1})\cdot(s - p_k)^r \tag{2}$$

The partial fraction expansion of $F(s)$ has the form

$$Y(s) = R_1/(s - p_1) + \cdots + R_{k-1}/(s - p_{k-1}) + R_{k1}/(s - p_k) + R_{k2}/(s - p_k)^2 + \cdots + R_{kr}/(s - p_k)^r \tag{3}$$

The coefficients R_i, $i = 1, \ldots, k - 1$ are given by Heaviside's rule, as also is R_{kr}, but the remaining $r - 1$ coefficients associated with the r-fold pole p_k are obtained by repeated differentiation:

$$R_{kj} = [1/(r - j)!]\cdot d^{r-j}/ds^{p-j}\, V(s)|_{s=pk} \quad (j = 1, 2, \ldots, r - 1) \tag{4}$$

where

$$V(s) = (s - p_k)^r Y(s) \tag{5}$$

Anybody who has ever used eqn (4) must have realised that there can be terrible algebraic tedium involved and that it can easily lead one into error.

In order to illustrate this, we write

$$V(s) = KN(s)/P(s) \tag{6}$$

with

$$P(s) = (s - p_1)(s - p_2)\cdots(s - p_{k-1}) \tag{7}$$

The normal rule for differentiation of a quotient (suppressing the argument s for simplicity) gives

$$\mathrm{d}V/\mathrm{d}s = K[\mathrm{d}N/\mathrm{d}s \cdot P - \mathrm{d}P/\mathrm{d}s \cdot N]/P^2 \tag{8}$$

and, after a little simplification,

$$\mathrm{d}^2V/\mathrm{d}s^2 = K[\{\mathrm{d}^2N/\mathrm{d}s^2 \cdot P - \mathrm{d}^2P/\mathrm{d}s^2 \cdot N\}P - 2\mathrm{d}P/\mathrm{d}s\{\mathrm{d}N/\mathrm{d}s \cdot P - \mathrm{d}P/\mathrm{d}s \cdot N\}]/P^3 \tag{9}$$

For the third derivative the algebraic expression becomes fairly horrendous, and yet two further stages would be needed in an example to be given below, which has a six-fold pole.

A helpful feature was pointed out many years ago,[3] but seems to have found its way into only one textbook.[4] That is the 'sum of residues' rule,

$$R_1 + R_2 + \cdots + R_{k-1} + R_{k1} = K\delta_{m,n-1} \tag{10}$$

where $\delta_{m,n-1}$ is the Kronecker delta (= 1 if $m = n - 1$, = 0 otherwise). Equation (10) shows that if the multiple pole is only twofold, no differentiation at all need be performed, since R_{k1} is determined once the other residues have been evaluated by Heaviside's rule.

The purpose of this note is to show that polynomial division and Heaviside's rule can be used to simplify enormously the process of repeatedly differentiating $V(s)$, in the case that there is only one multiple pole. Simplification is based on the observation that

$$V(s) = KN(s)/P(s) \tag{6}$$

is a rational function of numerator degree m and denominator degree $k - 1$. If its numerator is not of lower degree than the denominator (i.e., if $V(s)$ is an improper rational function), then, following polynomial division, it can be written as

$$V(s) = L(s) + T_1/(s - p_1) + T_2/(s - p_2) + \cdots + T_{k-1}/(s - p_{k-1}) \tag{11}$$

where $L(s)$ is a polynomial of degree $m - k + 1$. If $X(s)$ is a proper rational function, then L(s) is not present. In either case, the residues T_i are calculated by Heaviside's rule

$$T_i = (s - p_i)V(s)\big|_{s=pi} \tag{12}$$

In the form given in eqn (11), it is a very simple matter to differentiate $V(s)$ repeatedly:

$$\mathrm{d}^j V/\mathrm{d}v^j = \mathrm{d}^j L/\mathrm{d}v^j + (-1)^j j!\left[T_1/(s-p_1)^{j+1} \cdots + T_{k-1}/(s-p_{k-1})^{j+1}\right] \quad (13)$$

The successive derivatives of $L(s)$ are of progressively lower degree, and vanish altogether for $j \geq m - k + 1$.

An example

We consider an eigenvalue-assigning controller

$$C(s) = (2.068849s^3 + 6.681155s^2 + 5.559573s + 0.949219)/ [s(s^2 + 7s + 12.47461)] \quad (14)$$

for the unstable process

$$G(s) = 4(s+3)/[(s+1)(s-1)(s+2)] \quad (15)$$

along with the unity static gain reference input prefilter

$$F_l(s) = 0.23/(s+0.23) \quad (16)$$

The configuration in question is shown in Fig. 1.

This controller has been designed to place all six eigenvalues of the feedback loop at the location $s = -1.5$. Placing all eigenvalues at the same location – a generalisation of the concept of a critically damped system – is quite profound,[5] for it results in a system which has optimum stability with respect to every single parameter in process and controller, i.e., as any one parameter passes through its nominal value, all other parameters being held at theirs, the rightmost eigenvalue passes through a location ($s = -1.5$ here) which is as deep in the left half plane as possible.

The reference input filter is added to remove overshoot in the zero state step response of the process output, as shown in Fig. 2.

Subject to the above, the zero state unit step response of the process output has Laplace transform

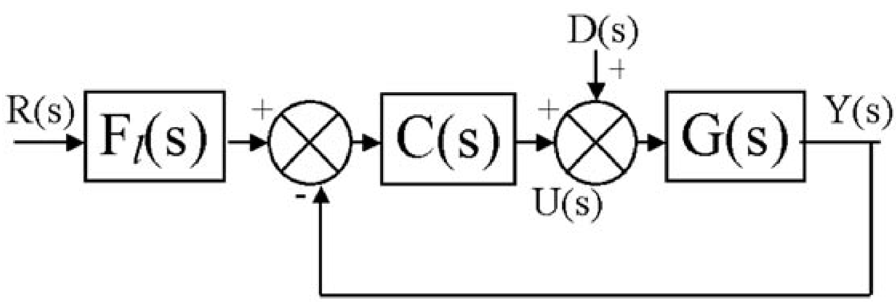

Fig. 1 *The feedback system considered.*

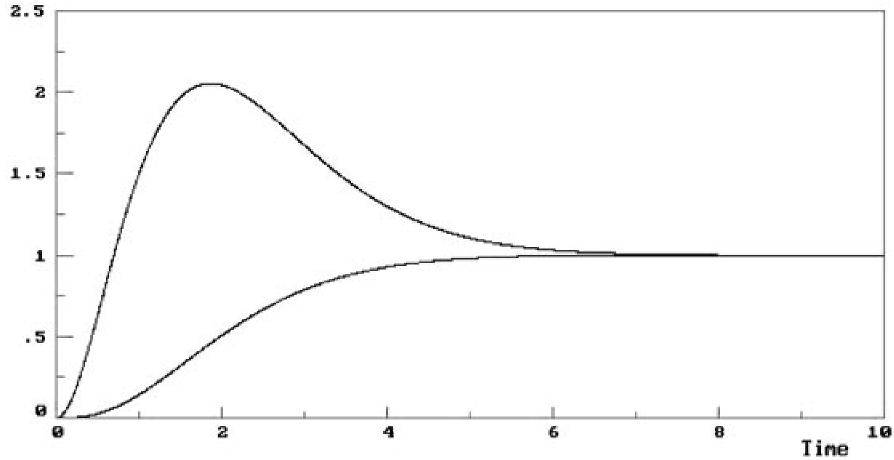

Fig. 2 *Zero state unit step response of process output in Fig. 1 (output, units vs time, s), with and without the reference input prefilter.*

$$Y(s) = \left(1.903341s^4 + 11.85669s^3 + 23.55479s^2 + 16.2177s + 2.619844\right) / \left[s(s+0.23)(s+1.5)^6\right] \quad (17)$$

This is a proper rational function with two simple poles and a pole of order six. The algebra involved in making a partial fraction expansion via the standard differentiation of a quotient route involves five successive differentiations of the improper rational function

$$V(s) = (s+1.5)^6 Y(s) = \left(1.903341s^4 + 11.85669s^3 + 23.55479s^2 | + 16.2177s + 2.619844\right) / \left(s^2 + 0.23s\right) \quad (18)$$

By polynomial division, using only the terms in the numerator as far as the vertical bar, and by using Heaviside's rule, eqn (18) is converted to the form

$$V(s) = 1.903341s^2 + 11.41892s + 20.92844 + 11.39063/s + 1.353316 \times 10^{-2} / (s+0.23) \quad (19)$$

The required derivatives are

$$\begin{aligned} dV/ds &= 3.806682s + 11.41892 - 11.39063/s^2 - 1.353316 \times 10^{-2}/(s+0.23)^2 \\ d^2V/ds^2 &= 3.806682 + 22.78126/s^3 + 2.706632 \times 10^{-2}/(s+0.23)^3 \\ d^3V/ds^3 &= -68.34378/s^4 - 8.119896 \times 10^{-2}/(s+0.23)^4 \\ d^4V/ds^4 &= 273.37512/s^5 + 3.2479584 \times 10^{-1}/(s+0.23)^5 \\ d^5V/ds^5 &= -1366.8756/s^6 - 1.6239792/(s+0.23)^6 \end{aligned} \quad (20)$$

We note that the algebra actually becomes simpler with each successive differentiation.

The partial fraction expansion of $Y(s)$ has the form

$$Y(s) = R_1/s + R_2/(s+0.23) + R_{31}/(s+1.5) + R_{32}/(s+1.5)^2 + R_{33}/(s+1.5)^3 + R_{34}/(s+1.5)^4 + R_{35}/(s+1.5)^5 + R_{36}/(s+1.5)^6 \quad (21)$$

where

$$\begin{aligned} R_1 &= sY(s)\big|_{s=0} = 1 \\ R_2 &= (s+0.23)Y(s)\big|_{s=-0.23} = 0.003225 \\ R_{31} &= [1/5!]\mathrm{d}^5V/\mathrm{d}s^5\big|_{s=-1.5} = -1.003225 \end{aligned} \quad (22)$$

The sum of residues rule[3] provides a check:

$$R_1 + R_2 + R_{31} = 0 \quad (23)$$

It is readily confirmed from eqns (4) and (20) that the final partial fraction expansion is

$$\begin{aligned} Y(s) = {} & 1/s + 0.003225/(s+0.23) - 1.003225/(s+1.5) - 1.504096/(s+1.5)^2 \\ & - 2.255202/(s+1.5)^3 + 1.478266/(s+1.5)^4 + 0.6380075/(s+1.5)^5 \\ & + 0.4781662/(s+1.5)^6 \end{aligned} \quad (24)$$

Discussion

A suggestion, which the author believes to be new, has been made and illustrated to form the partial fraction expansion of a proper rational function with a single pole of high multiplicity, by simplifying the process of repeated differentiation, through polynomial division and Heaviside's rule. The great beauty of the idea is that the algebra actually becomes simpler with each successive differentiation. Unfortunately, it does not seem to be adaptable *in general* to the case of several high order poles, or to multiple complex poles. (With regard to the latter, there are possibly helpful comments in Ref. [3]). However, if there is a double real pole and another higher order pole, the former may be incorporated into $V(s)$ and that can then be expanded into partial fractions using Heaviside's rule augmented by the sum of residues rule.

It is certainly true that partial fraction expansions of the type considered are usually effected nowadays at a keystroke, using computer-aided design and analysis packages. The author is strongly of the opinion, however, that basic analytical skills should not be allowed to atrophy and that one should always be able to check the results provided by packages, especially when, as is sometimes alarmingly the case, familiarity with system behaviour alerts one to the fact that packages can err.[6]

Acknowledgement

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