Thevenin and Norton's theorems: powerful pedagogical tools for treating special cases of electric circuits

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Abstract The importance of Thevenin's and Norton's theorems when dealing with special cases of electric circuits is demonstrated. These cases have typically been dealt with in difficult and non-systematic ways and as a result, pedagogical effectiveness is minimised and students are reluctant to tackle them. These theorems are powerful pedagogical tools for the solution of many problems in circuit analysis.

Keywords equivalent inductance; input impedance; Norton's theorem; operational amplifier circuits; resonance frequency; Thevenin's theorem; two-port circuits

Thevenin's theorem and Norton's theorem are among the most important theorems in electric circuits, especially when interest is focused on a particular part of the circuit (i.e. finding the voltage, current or power in an impedance) or when dealing with problems of load matching. For a linear circuit in sinusoidal steady state these are as follows:

> *A given linear circuit in sinusoidal steady state with two terminals A and B may be replaced (after its transformation to the frequency domain) by an independent voltage source in series with an impedance (Thevenin's theorem) or by an independent current source in parallel with the same impedance (Norton' s theorem), with the assumption that all sources are of the same frequency.*

Thevenin's theorem and Norton's theorem are shown schematically in Fig. 1. If the circuit contains sources with different frequencies, the equivalent Thevenin and Norton circuits must be calculated separately for each frequency.[1,2] Thus, for each frequency, separate partial equivalent circuits result, and not a single circuit with source and impedance, since in this case, in the frequency domain the impedances correspond to different frequencies and cannot be added.

The natural meaning of the parameters appearing in the equivalent Thevenin and Norton circuits are:

V_{TH}: open circuit voltage of the linear circuit at the A and B terminals.
I_N: short circuit current of the linear circuit through the A and B terminals.
$Z_{TH} = Z_N$: impedance of the linear circuit at the A and B terminals.

These three parameters are related by the following relationship:

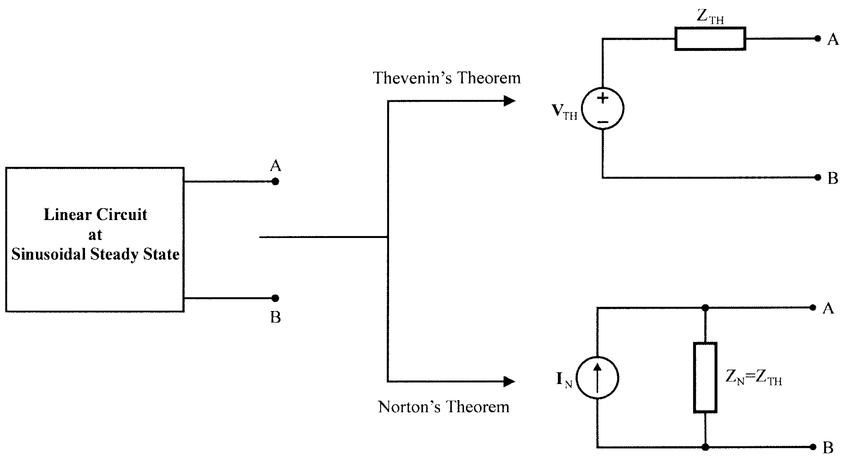

Fig. 1 *Thevenin's and Norton's theorems.*

$$Z_{TH} = Z_N = \frac{V_{TH}}{I_N} \tag{1}$$

The analysis and procedure followed to calculate the parameters of the equivalent Thevenin and Norton circuits depend on the kind of sources included in the linear circuit. Thus, the following cases are considered:

(a) The circuit contains only independent sources[1–6]

1. To find V_{TH}, the open circuit voltage at the A and B terminals is calculated by applying known techniques (mesh-current method, node-voltage method, voltage divider, current divider, superposition, etc.).
2. To find Z_{TH}, all sources are zeroed (voltage sources are short circuited and current sources are open circuited) and the total impedance of the circuit between the A and B terminals is calculated.
3. To find I_N, terminals A and B are short circuited and the current through this short circuit is calculated by applying the known techniques previously referred to.
4. To find Z_N, the procedure as for Z_{TH} is followed since $Z_{TH} = Z_N$.

(b) The circuit contains independent and dependent sources[1–3,6]

1. V_{TH} is calculated as in case (a).
2. I_N is calculated as in case (a).
3. The impedance Z_{TH} (or Z_N) is calculated using relationship (1).

Note

The impedance Z_{TH} (or Z_N) cannot be calculated as in case (a) because the circuit cannot be neutralised due to the existence of the dependent sources.

(c) The circuit contains only dependent sources[1,2,6]

When there are no independent sources, obviously $V_{TH} = 0$ and $I_N = 0$. Therefore, finding the equivalent Thevenin and Norton circuits reduces to calculation of the impedance $Z_{TH} = Z_N$. This is carried out as follows:

An independent current source of $1\angle 0°$ A is connected at the A and B terminals and the voltage V_x at its terminals is calculated using known techniques. However, since $V_x = Z_{TH} \cdot 1\angle 0°$, it is concluded that:

$$Z_{TH} = V_x, \quad \text{in resistance units } (\Omega) \tag{2}$$

In this paper, the importance of these theorems is presented (further to known applications such as, for example, matching problems) for special cases of electric circuits that are currently dealt with in difficult and non-systematic ways, such that the pedagogical effectiveness is minimised and the students are reluctant to attempt them.

Special cases

Special case A: operational amplifier (op amp) circuits with a complicated input circuit applied to the inverting or non-inverting terminal

Many op amp circuits are designed to perform a specific operation (i.e. digital-to-analogue converter, or DAC), and have a complicated input circuit connected to the inverting or non-inverting input terminal. Determining the op amp output for such devices is very difficult by conventional methods, i.e. by applying Kirchhoff's current law to some nodes.

A systematic and therefore very pedagogical solution method is to cut off the complicated input circuit between ground and the inverting (or non-inverting) input terminal and to replace it by an equivalent Thevenin circuit according to case (a) or (b) described above. This results in a known equivalent op amp circuit (i.e. an inverting amplifier), the analysis of which is known to the students. Thus, finding the output of the complicated op amp circuit is reduced to finding the output of a known, simple op amp circuit, in which the Thevenin equivalent circuit of the complicated input part is connected between ground and the inverting (or non-inverting) input terminal.

Example 1

Show that the circuit of Fig. 2 is a four-bit DAC, and has an analogue output voltage

$$v_0 = -\frac{R_f}{R}\left(\frac{v_1}{2} + \frac{v_2}{4} + \frac{v_3}{8} + \frac{v_4}{16}\right)$$

Next, for $R_f = 6\,\text{K}\Omega$ and $R = 5\,\text{K}\Omega$ calculate the output, when the digital input is $[v_1 v_2 v_3 v_4] = [1101]$.

Solution The part of the circuit to the left of the section $t - t'$ (between ground and the inverting terminal) is cut off and its Thevenin equivalent circuit found (Fig. 3), according to case (a).

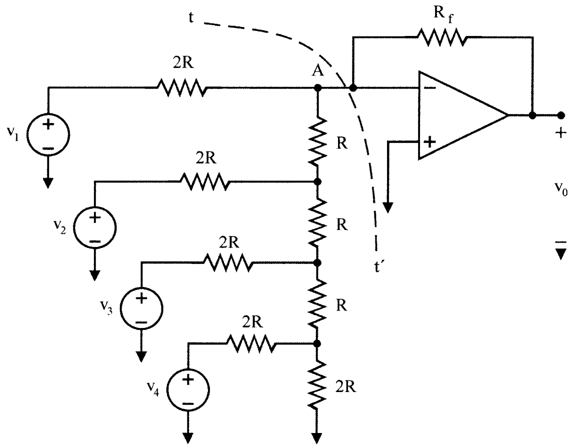

Fig. 2 *Four-bit DAC.*

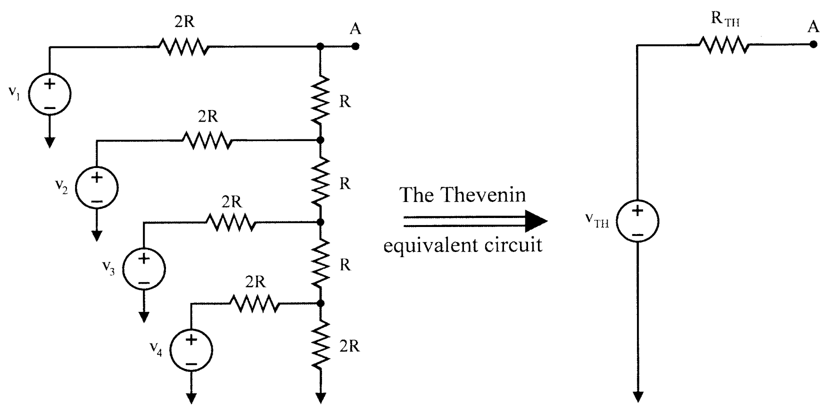

Fig. 3 *Finding equivalent Thevenin circuit.*

In order to find R_{TH} (d.c. circuit) the voltage sources of the cut off circuit are short circuited and by applying the rules of resistance combinations:

$$R_{TH} = [[[2R//2R + R]//2R + R]//2R + R]//2R = R \Rightarrow R_{TH} = R$$

For v_{TH} calculation, the mesh current method is applied to the circuit of Fig. 4. The mesh current equations in matrix form are:

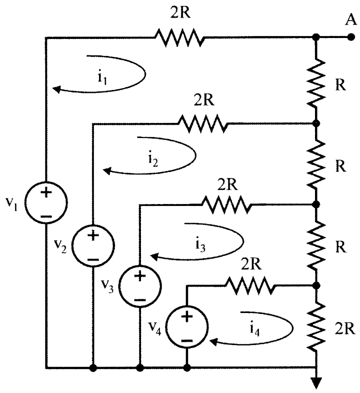

Fig. 4 *Mesh current method.*

$$R\cdot\begin{bmatrix}5 & -2 & 0 & 0\\ -2 & 5 & -2 & 0\\ 0 & -2 & 5 & -2\\ 0 & 0 & -2 & 4\end{bmatrix}\cdot\begin{bmatrix}i_1\\ i_2\\ i_3\\ i_4\end{bmatrix}=\begin{bmatrix}v_1-v_2\\ v_2-v_3\\ v_3-v_4\\ v_4\end{bmatrix}=\begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}i_1\\ i_2\\ i_3\\ i_4\end{bmatrix}=\frac{1}{R}\cdot\begin{bmatrix}0.25 & -0.125 & -0.0625 & -0.03125\\ 0.125 & 0.1875 & -0.15625 & -0.078125\\ 0.0625 & 0.09375 & 0.171875 & -0.1640625\\ 0.03125 & 0.046875 & 0.0859375 & 0.16796875\end{bmatrix}\cdot\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}$$

Thus,

$$v_{\text{TH}}=R\cdot i_1+R\cdot i_2+R\cdot i_3+2R\cdot i_4=R\cdot[1\quad 1\quad 1\quad 2]\cdot\begin{bmatrix}i_1\\ i_2\\ i_3\\ i_4\end{bmatrix}$$

$$=R\cdot[1\quad 1\quad 1\quad 2]\cdot\frac{1}{R}\cdot\begin{bmatrix}0.25 & -0.125 & -0.0625 & -0.03125\\ 0.125 & 0.1875 & -0.15625 & -0.078125\\ 0.0625 & 0.09375 & 0.171875 & -0.1640625\\ 0.03125 & 0.046875 & 0.0859375 & 0.16796875\end{bmatrix}\cdot\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}$$

$$\Rightarrow v_{\text{TH}}=[0.5\quad 0.25\quad 0.125\quad 0.0625]\cdot\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}\Rightarrow v_{\text{TH}}=\frac{v_1}{2}+\frac{v_2}{4}+\frac{v_3}{8}+\frac{v_4}{16}$$

Thus, the equivalent circuit of the op amp is as shown in Fig. 5.

This is a standard inverting amplifier and therefore the output voltage is given by:

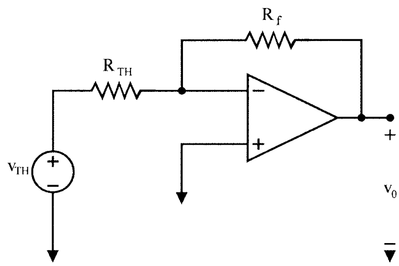

Fig. 5 *Op amp equivalent circuit.*

$$v_0 = -\frac{R_f}{R_{TH}} v_{TH} = -\frac{R_f}{R}\left(\frac{v_1}{2}+\frac{v_2}{4}+\frac{v_3}{8}+\frac{v_4}{16}\right)$$

Thus, the initial circuit is a four-bit DAC, since each bit has half the weight of the next higher bit.

Finally, for $R_f = 6\,K\Omega$ and $R = 5\,K\Omega$, the analogue output for the given digital input $[v_1 v_2 v_3 v_4] = [1101]$ is:

$$v_0 = -\frac{R_f}{R}\left(\frac{v_1}{2}+\frac{v_2}{4}+\frac{v_3}{8}+\frac{v_4}{16}\right) = -\frac{6}{5}\left(\frac{1}{2}+\frac{1}{4}+\frac{0}{8}+\frac{1}{16}\right) \Rightarrow v_0 = -0.975\,\text{V}$$

Special case B: finding the equivalent inductance of mutually coupled coils without ohmic losses

A mutually coupled coil without losses is, in fact, a circuit with inductances and dependent voltage sources, because the voltage of each coil is equal to the algebraic sum of its induced voltage and the mutual voltage due to the other coil. However, the voltages due to the mutual inductances depend on the other coil currents and therefore each coupled coil is equivalently replaced by its impedance $j\omega L$ and by dependent voltage sources due the coupling with the other coils.[1]

Thus, finding the equivalent inductance is impossible by simply applying the combination rules of ideal coils due to the existence of dependent voltage sources. Methods to deal with such cases (working in the time domain) are difficult and time consuming and do not provide students with a unique and systematic solution to meet all the possible combinations that they might meet.

These difficulties cease to exist if the equivalent inductance is found in a way similar to that followed when finding Z_{TH} in a circuit that includes only dependent sources (case (c)). So, an independent current source $1\angle 0°$ A at $\omega = 1$ rad/s (or at any other frequency) is connected to the A and B terminals of the coupled coils circuit and the voltage V_x at the current source terminals is calculated using known techniques.

However, since this circuit has purely inductive behaviour, the voltage V_x will lead the source current by 90°, that is:

$$V_x = \rho\,\angle 90°\ \text{V}$$

But,

$$V_x = Z_{AB} \cdot I \Rightarrow \rho\angle 90^\circ = j1 \cdot L_{eq} \cdot (1\angle 0^\circ) = L_{eq}\angle 90^\circ \Rightarrow L_{eq} = \rho \qquad (3)$$

Therefore, the equivalent inductance of the coupled coils circuit (without ohmic losses) has a value equal to the amplitude of the voltage V_x in inductance units, that is in henries (H).

Note

Instead of $\omega = 1$ rad/s, if a random frequency is selected, then relationship (3) takes the form $L_{eq} = \dfrac{\rho}{\omega}$. Therefore, the selection of unit frequency is proposed, since the equivalent inductance is then calculated directly.

Example 2

Calculate the equivalent inductance of the circuit shown in Fig. 6.

Solution Selecting $\omega = 1$ rad/s and simultaneously placing an independent current source $1\angle 0^\circ$ A between terminals A and B, the circuit in the frequency domain takes the form shown in Fig. 7.

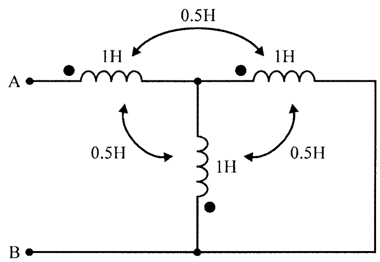

Fig. 6 *Circuit for Example 2.*

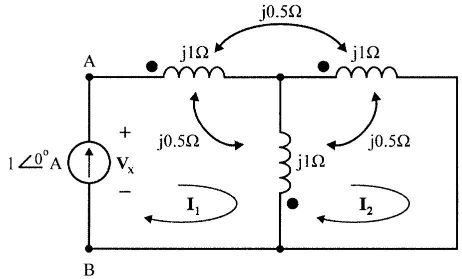

Fig. 7 *Example 2 frequency domain circuit.*

The mesh current equations in matrix form are:

$$\begin{bmatrix} j1 + j1 - 2\cdot j0.5 & -j1 + j0.5 - j0.5 + j0.5 \\ -j1 + j0.5 - j0.5 + j0.5 & j1 + j1 + 2\cdot j0.5 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_x \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} j1 & -j0.5 \\ -j0.5 & j3 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_x \\ 0 \end{bmatrix}$$

Substituting the first line of the system by the equation $I_1 = 1\angle 0°$ A, which applies to the current source, the following algebraically equivalent system results:

$$\begin{bmatrix} 1 & 0 \\ -j0.5 & j3 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1\angle 0° \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} I_1 = 1 + j0\,\text{A} \\ I_2 = 0.167 + j0\,\text{A} \end{matrix}$$

From the first line of the initial system we obtain:

$$V_x = j1\cdot I_1 - j0.5\cdot I_2 = j1\cdot(1 + j0) - j0.5\cdot(0.167 + j0) = 0.917\angle 90°\ \text{V}$$

Therefore, according to relationship (3), the equivalent inductance is $L_{eq} = 0.917\,\text{H}$

Special case C: finding the input impedance of a circuit including passive elements *R*, *L*, *C* and *M*

A circuit including passive elements *R*, *L*, *C* and *M* is in fact a circuit with impedances R, $j\omega L$, $\dfrac{-j}{\omega C}$ and dependent voltage sources, because the voltage of each coil is equal to the algebraic sum of its induced voltage and the mutual voltages due to the inductances of the other coils. So, it is impossible to find the input impedance by applying the rules of impedance combinations due to the existence of the dependent sources.

However, it is possible to determine the impedance in a way similar to that followed in order to find Z_{TH} in a circuit containing only dependent sources (case (c)). An independent current source of $1\angle 0°$ A at an angular frequency ω is connected to the A and B terminals of the circuit, and the voltage V_x at the current source terminals is calculated using known techniques. But,

$$V_x = Z_{in}\cdot I \Rightarrow V_x = Z_{in}\cdot(1\angle 0°) \Rightarrow Z_{in} = V_x,\ \text{in resistance units}\ (\Omega) \tag{4}$$

therefore, the circuit input impedance is equal to the voltage V_x in resistance units, that is in ohms (Ω).

Note If

- Im (V_x) > 0, the circuit presents inductive behaviour.
- Im (V_x) < 0, the circuit presents capacitive behaviour.

Example 3

Calculate the input impedance of the circuit shown in Fig. 8.

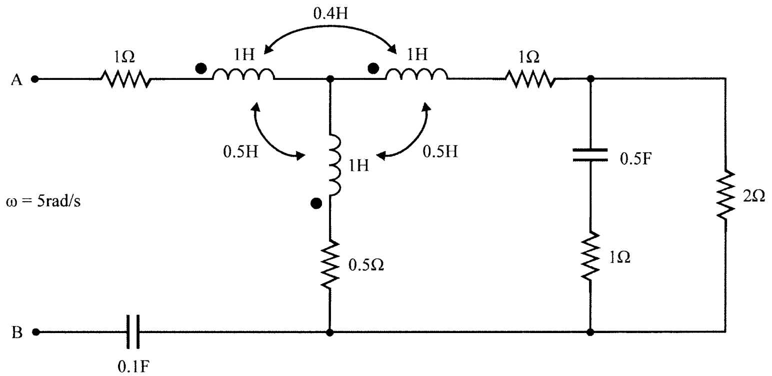

Fig. 8 *Circuit for Example 3.*

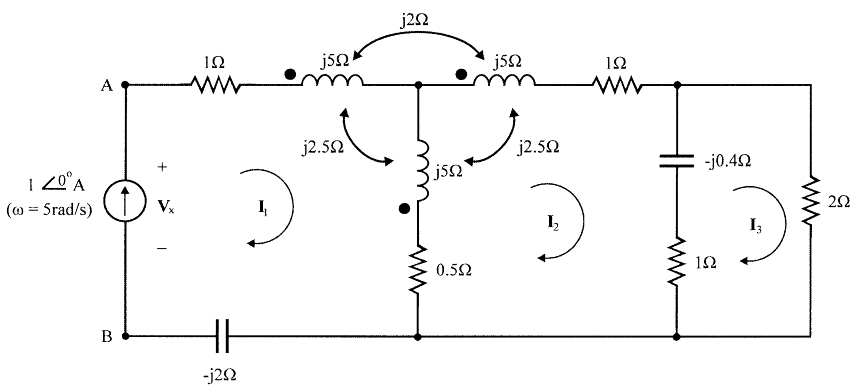

Fig. 9 *Example 3 frequency domain circuit.*

Solution Transforming the circuit to the frequency domain at $\omega = 5$ rad/s and simultaneously placing an independent current source of $1\angle 0°$ A (at the same frequency) at the A and B terminals, the circuit takes the form shown in Fig. 9.

The mesh current equations in matrix form are:

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} \Sigma V_1 \\ \Sigma V_2 \\ \Sigma V_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1.5+j3 & -0.5-j3 & 0 \\ -0.5-j3 & 2.5+j14.6 & -1+j0.4 \\ 0 & -1+j0.4 & 3-j0.4 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_x \\ 0 \\ 0 \end{bmatrix}$$

Setting $I_1 = 1\angle 0°$ A in the first line of the system equation, which applies to the current source, the resulting algebraically equivalent system is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5-j3 & 2.5+j14.6 & -1+j0.4 \\ 0 & -1+j0.4 & 3-j0.4 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \cdot \begin{bmatrix} 1\angle 0° \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} I_1 = 1 + j0\,\text{A} \\ I_2 = 0.20291 - j0.00375\,\text{A} \end{matrix}$$

From the first line of the initial system we obtain:

$$V_x = (1.5 + j3)\cdot I_1 + (-0.5 - j3)\cdot I_2 = 1.38729 + j2.39314\,\text{V}$$

The circuit presents inductive behaviour since Im (V_x) > 0, and according to relationship (4), the input impedance of the given circuit is given by $Z_{in} = 1.38729 + j2.39314\,\Omega$.

Special case D: finding the resonance frequency of a circuit including passive elements *R*, *L*, *C* and *M* or passive elements *R*, *L*, *C* and dependent sources

In order to find the resonance frequency for a circuit consisting of passive elements *R*, *L*, *C* and *M*, as for case (c), the circuit input impedance is found after connecting an independent current source of $1\angle 0°$ A at an angular frequency ω between the A and B terminals. Solution of the equation Im (Z_{in}) = 0 (if it exists) gives the resonant frequency (or frequencies) of the circuit.

The resonant frequency of circuits consisting of elements *R*, *L*, *C* and dependent sources is found in exactly the same way.

Example 4

Calculate the resonant frequency of the circuit shown in Fig. 10.

Solution Transforming the circuit to the frequency domain at an angular frequency ω and simultaneously placing an independent current source of $1\angle 0°$ A between the A and B terminals, the circuit takes the form shown in Fig. 11.

The mesh current equations in matrix form are:

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \Sigma V_1 \\ \Sigma V_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 - j\dfrac{10}{\omega} + j2\omega & 0 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_x - 3V_0 \\ 3V_0 \end{bmatrix}$$

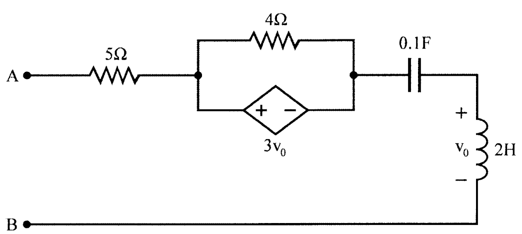

Fig. 10 *Circuit for Example 4.*

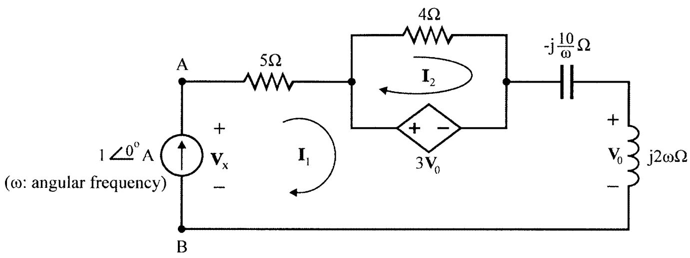

Fig. 11 *Example 4 frequency domain circuit.*

Setting $I_1 = 1\angle 0°$ A in the first line of the system equation, which applies to the current source, the resulting algebraically equivalent system is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1\angle 0° \\ 3V_0 \end{bmatrix} = \begin{bmatrix} 1\angle 0° \\ j6\omega I_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ -j6\omega & 4 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1\angle 0° \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} I_1 = 1\angle 0° \text{ A} \\ I_2 = 1.5\omega\angle 90° \text{ A} \end{matrix}$$

From the first line of the initial system we obtain:

$$\left(5 - j\frac{10}{\omega} + j2\omega\right) \cdot I_1 = V_x - 3V_0 \Rightarrow V_x = 3V_0 + \left(5 - j\frac{10}{\omega} + j2\omega\right) \cdot I_1$$

$$\Rightarrow V_x = j6\omega \cdot I_1 + \left(5 - j\frac{10}{\omega} + j2\omega\right) \cdot I_1 = \left(j6\omega + 5 - j\frac{10}{\omega} + j2\omega\right) \cdot 1\angle 0°$$

$$\Rightarrow V_x = 5 + j\left(8\omega - \frac{10}{\omega}\right) \text{V}$$

Therefore, according to relationship (4), the input impedance of the given circuit is given by:

$$Z_{in} = 5 + j\left(8\omega - \frac{10}{\omega}\right) \Omega$$

Finally, satisfying the resonance condition gives:

$$\text{Im}(Z_{in}) = 0 \Rightarrow 8\omega - \frac{10}{\omega} = 0 \Rightarrow 8\omega^2 = 10 \Rightarrow \omega^2 = 1.25 \overset{\omega>0}{\Rightarrow} \omega = 1.118 \text{ rad/s}$$

Thus, the resonant frequency of the circuit is $\omega_0 = 1.118$ rad/s.

Finding matrices describing any two-port circuit and especially circuits including passive elements *R*, *L*, *C* and *M* or *R*, *L*, *C* and dependent sources

It is known that in a two-port circuit (Fig. 12) the only variables of interest are the voltages and currents at the terminals (V_1, V_2, I_1, I_2). These terminal quantities, com-

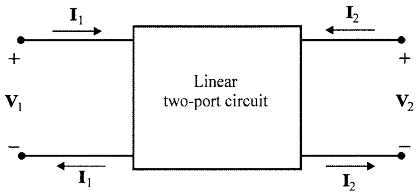

Fig. 12 *Two-port circuit.*

bined in pairs, give six different ways to describe a two-port circuit. These descriptions are termed impedance parameters (or z-parameters); admittance parameters (or y-parameters); hybrid parameters (or h-parameters); inverse hybrid parameters (or g-parameters); transmission parameters (or a-parameters); and inverse transmission parameters (or b-parameters).[1–4]

Assuming that z-parameter calculations are desired for a two-port circuit, the two-port circuit is described by the following matrix expression of input and output variables:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

where

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \quad \text{: open-circuit input impedance}$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \quad \text{: open-circuit transfer impedance from port 1 to port 2}$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} \quad \text{: open-circuit transfer impedance from port 2 to port 1}$$

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} \quad \text{: open-circuit output impedance}$$

Since all the z-parameters are ratios of two input and/or output variables and their values finally depend only on the internal structure of the two-port circuit, usually they are calculated by applying the rules of impedance combinations and by using voltage and current dividers. However, this is not possible when the two-port circuit includes a combination of all the passive elements R, L, C and M because the rules for impedance combinations are then not valid, for the reasons referred to when discussing special case C.

The same considerations hold for two-port circuits with elements R, L, C and dependent sources. In these cases, z-parameter calculations may be performed as follows:

- In order to find the parameters z_{11} and z_{21} the two-port circuit is activated at port 1 by a current source of $1\angle 0°$ A and simultaneously the condition $I_2 = 0$ is satisfied. The voltage at the current source terminals is calculated, and is numerically equal to the parameter z_{11}, in resistance units, as in special cases C and D. Next, calculating the voltage V_2, the parameter z_{21} is readily available, because from its definition, and given that $I_1 = 1$ A, it is obvious that it is numerically equal to this voltage value in resistance units.
- For the parameters z_{12} and z_{22} the two-port circuit is activated at port 2 by a current source of $1\angle 0°$ A and simultaneously the condition $I_1 = 0$ is satisfied. The voltage at the current source terminals is calculated, and is numerically equal to the parameter z_{22}, in resistance units, as in special cases C and D. Next, calculating the voltage V_1, the parameter z_{12} is readily available, because from its definition, and given that $I_2 = 1$ A, it is obvious that it is numerically equal to this voltage value in resistance units.

Example 5
Find the z-parameters for the two-port circuit shown in Fig. 13 at $f = 60$ Hz.

Solution Calculation of z_{11} and z_{21}:

Transforming the circuit to the frequency domain, activating port 1 of the two-port circuit with a current source of $1\angle 0°$ A, and simultaneously satisfying the condition $I_2 = 0$, the circuit takes the form shown in Fig. 14.

The mesh current equations in matrix form are:

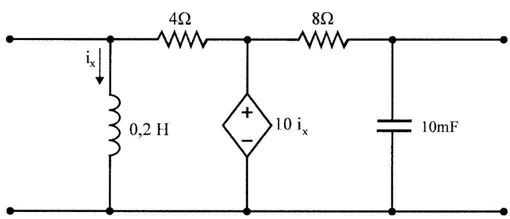

Fig. 13 *Circuit for Example 5.*

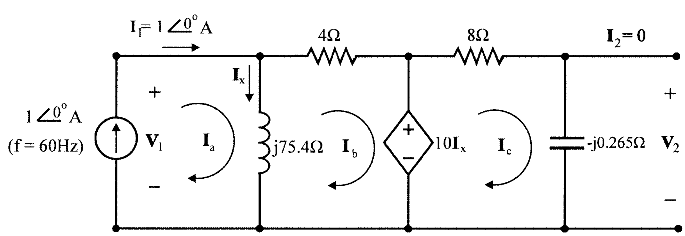

Fig. 14 *Example 5 frequency domain circuit: activating port 1.*

$$\begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} \Sigma V_a \\ \Sigma V_b \\ \Sigma V_c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} j75.4 & -j75.4 & 0 \\ -j75.4 & 4+j75.4 & 0 \\ 0 & 0 & 8-j0.265 \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} V_1 \\ -10I_x \\ 10I_x \end{bmatrix}$$

Setting $I_\alpha = 1\angle 0°$ A in the first line of the system equation, which applies to the current source, the resulting algebraically equivalent system is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -j75.4 & 4+j75.4 & 0 \\ 0 & 0 & 8-j0.265 \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1\angle 0° \\ -10I_x \\ 10I_x \end{bmatrix} = \begin{bmatrix} 1\angle 0° \\ -10(I_a - I_b) \\ 10(I_a - I_b) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 10-j75.4 & -6+j75.4 & 0 \\ -10 & 10 & 8-j0.265 \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1\angle 0° \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} & I_a = 1 + j0\,\text{A} \\ \Rightarrow\ & I_b = 1.00419 + j0.05272\,\text{A} \\ & I_c = -0.00306 - j0.065997\,\text{A} \end{aligned}$$

From the first line of the initial system we obtain:

$$V_1 = j75.4 \cdot I_a - j75.4 \cdot I_b = 3.9876\angle -4.5°\ \text{V}$$

Also, from Fig. 14 we obtain:

$$V_2 = (-j0.265) \cdot I_c = 0.0175\angle 177.35°\ \text{V}$$

Thus,

$$z_{11} = 3.9876\angle -4.5°\ \Omega \quad \text{and} \quad z_{21} = 0.0175\angle 177.35°\ \Omega$$

Calculation of the z_{12} and z_{22}:

Transforming the circuit to the frequency domain, activating port 2 of the two-port circuit with a current source of $1\angle 0°$ A, and simultaneously satisfying the condition $I_1 = 0$, the circuit takes the form shown in Fig. 15.

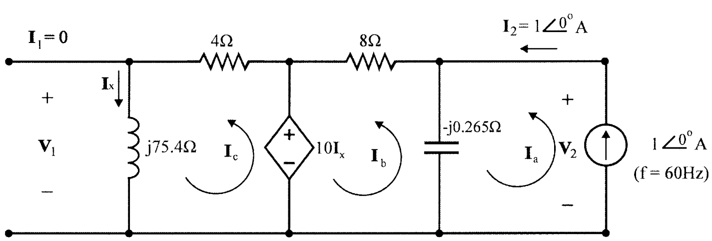

Fig. 15 *Example 5 frequency domain circuit: activating port 2.*

The mesh current equations in matrix form are:

$$\begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} \Sigma V_a \\ \Sigma V_b \\ \Sigma V_c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -j0.265 & j0.265 & 0 \\ j0.265 & 8-j0.265 & 0 \\ 0 & 0 & 4+j75.4 \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} V_2 \\ -10I_x \\ 10I_x \end{bmatrix}$$

Setting $I_\alpha = 1\angle 0°$ A in the first line of the system equation, which applies to the current source, the resulting algebraically equivalent system is:

$$\begin{bmatrix} 1 & 0 & 0 \\ j0.265 & 8-j0.265 & 0 \\ 0 & 0 & 4+j75.4 \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1\angle 0° \\ -10I_x \\ 10I_x \end{bmatrix} = \begin{bmatrix} 1\angle 0° \\ -10I_c \\ 10I_c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ j0.265 & 8-j0.265 & 10 \\ 0 & 0 & -6+j75.4 \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1\angle 0° \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} & I_a = 1 + j0\,\text{A} \\ \Rightarrow\ & I_b = 0.001096 - j0.033088\,\text{A} \\ & I_c = 0 + j0\,\text{A} \end{aligned}$$

From the first line of the initial system we obtain:

$$V_2 = -j0.265 \cdot I_a + j0.265 \cdot I_b = 0.2651\angle -88.1°\ \text{V}$$

Also, from Fig. 15 we obtain:

$$V_1 = (j75.4) \cdot I_c = 0\,\text{V}$$

Thus,

$$z_{22} = 0.2651\angle -88.1°\ \Omega \quad \text{and} \quad z_{12} = 0\,\Omega$$

Conclusions

The ability of a method to inspire self-confidence in students in order to effectively solve problems that were previously considered too difficult, is the key to success during students' education and development.

Extension of the use of Thevenin's theorem and Norton's theorem beyond the usual applications (i.e. matching problems), as described in this paper, shows their importance as pedagogical tools for the simple and systematic solution of many other problem categories that students meet during studies of electrical circuits.

Thus, when faced with op amp circuits designed to execute specific functions, in which a complicated input circuit is connected to the inverting terminal or non-inverting terminal, the student has the necessary analysis tools. Also, by applying a similar technique to that used to find Thevenin or Norton equivalent circuits to circuits that include only passive elements and dependent sources (case (c)), students can solve problems dealing with finding the equivalent inductance, input impedance and resonant frequency of complicated circuits. These types of problem previously may well have proven very difficult for students, causing them to lose self-confidence.

The method is also applicable to any two-port circuit, enabling students to rapidly determine all network parameters.

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