A case study to motivate engineering students to do mathematical proofs

Rodney G. Roberts
Department of Electrical and Computer Engineering, Florida A&M University-Florida State University, Tallahassee, Florida, USA
E-mail: rroberts@eng.fsu.edu

Abstract Although engineering students are trained to solve a wide range of problems, they often lack experience in doing mathematical proofs. To encourage students to learn this important skill, the author presents several proofs of a well-known identity. The methods include simple algebraic and geometric proofs, engineering mathematical solutions, and techniques from finite mathematics.

Keywords arithmetic series; engineering education; mathematics; proofs

Although students are exposed to a significant amount of mathematics in an undergraduate engineering curriculum, students often do not learn the art of doing mathematical proofs. Undergraduate engineering students generally take courses in calculus, differential equations, linear algebra, and transform theory. While these courses equip students with the necessary tools to solve a wide range of problems, the method and value of mathematical proofs are generally not emphasised.

However, there are important advantages for students to have at least some exposure to mathematical proofs. A better understanding of how to prove a result can greatly enhance an engineering student's ability to learn. A student who knows how to prove a result is less likely to make mistakes using that result and is more likely to catch misremembered formulae, even if the student has only a rudimentary understanding of what mathematical rigour is. Exposure to proofs is particularly important for students going on to graduate school. While a working knowledge of mathematical techniques is sufficient for most students, students going into research will need to learn how to prove their results if they want to publish, particularly if they enter more analytical areas such as automatic control, communication systems, or signal processing. Students who need a stronger analytical background will most likely take courses where they will learn how to formally prove results. However, it is a good idea for them to become exposed to proofs at an earlier stage even if it is done only informally.

Instructors can occasionally introduce students to techniques such as mathematical induction, proof by contradiction, contraposition, and other methods during a lecture or a seminar. The problem is to motivate students to continue to develop these skills outside the classroom. One approach is to present students with a case study where a result is proven using various techniques. Once students see that a result can be proven in more than one way, they will find that they have plenty of opportunity to practise their skills. In the process they will learn the advantages and limitations of different approaches. In this article the author presents a number of alternative proofs of a simple well-known identity. The methods of proof include simple algebraic and geometric proofs, engineering mathematical solutions, and techniques from abstract algebra and finite mathematics.

This paper is somewhat motivated by an interesting book by Elisha S. Loomis called *The Pythagorean Proposition* that contains around 370 proofs of the Pythagorean Theorem.[3] Loomis divides these proofs into four categories: algebraic proofs (e.g., similar triangles), geometric proofs (comparisons of areas), vector-based proofs, and dynamic proofs (based on mass, velocity, and force). The goal of this paper is more modest. The next section contains 20 proofs of the simple identity

$$S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{1}$$

The proofs were chosen based on their appeal to engineering students. Some of the proofs have advantages over the others, e.g., some are simpler and more elegant while others lead to more general results. At the end of the next section, students are invited to prove a similar identity using modifications of as many of these proofs as they can. This will allow them some practice and will hopefully motivate them to come up with their own examples. The subsequent section contains some examples of incorrect proofs and an explanation of why they did not work. The conclusions of the article are then given.

Twenty proofs of a well-known identity

Proof by induction

Proof by induction is one of the most important tools for proving mathematical results. Basically, proof by induction proceeds as follows. One first shows that a statement P_n is true for $n = 1$. Next, one shows that for $n \geq 1$, the statement that P_n is true implies that P_{n+1} is true. Mathematical induction then states that P_n is true for all $n \geq 1$. Equation (1) is clearly true for $n = 1$. Assume that it is true for n. Then

$$S_{n+1} = S_n + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}, \tag{2}$$

proving the desired result.

Although it is a powerful and completely rigorous technique for proving results, mathematical induction often does not add insight into a result. The next nineteen proofs are generally not as rigorous as mathematical induction, but they do provide some insight into the identity. The next three proofs are based on arithmetic. The first is perhaps the simplest of all the proofs of eqn (1). Those familiar with mathematical history may recall the story of how Gauss came up with this derivation as a young boy.[1]

Arithmetic approach

Write the summation twice, once in ascending order and then, immediately underneath, in descending order as shown:

$$\begin{array}{ccccccc} 1 & + & 2 & + & \cdots & + & n \\ n & + & n-1 & + & \cdots & + & 1 \\ \hline (n+1) & + & (n+1) & + & \cdots & + & (n+1) \end{array}.$$

This results in the number $n + 1$ occurring n times, giving a sum of $n(n + 1)$. Since the summation appears twice, we divide by two to obtain eqn (1).

Summation trick

Consider the following summation argument:

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n} (n+1-k) = \sum_{k=1}^{n} (n+1) - \sum_{k=1}^{n} k = n(n+1) - \sum_{k=1}^{n} k. \tag{3}$$

The first equality in (3) is just the statement that reversing the order in a summation does not change the sum. Regrouping terms yields $2\Sigma_{k=1}^{n} k = n(n + 1)$ giving us the desired result.

Another summation trick

The square of a sum is equal to the sum of the squares plus twice the cross terms:

$$\left(\sum_{k=0}^{n} x_k\right)^2 = \sum_{k=0}^{n} x_k^2 + 2\sum_{0 \le i < j \le n} x_i x_j = \sum_{k=0}^{n} x_k^2 + 2\sum_{j=1}^{n}\sum_{i=0}^{j-1} x_i x_j. \tag{4}$$

Setting $x_i = 1$, $i = 0, 1, \ldots, n$ and noting that $\Sigma_{i=0}^{j-1} 1 = j$ yields $(n + 1)^2 = n + 1 + 2S_n$. Simplifying gives the desired result.

The next three proofs are based on simple geometry and should appeal to the more visually inclined student.

An area-based approach

Consider a group of squares in the plane each with an area of 1 square metre arranged in n columns with one square in the first column, two squares in the second column, and so on with n squares in the last. Suppose these squares are glued together to form a jagged triangular figure. The area of this figure is S_n square metres. If we take an exact copy of this figure, rotate it by 180°, and fit it together with the first figure, we will form an $n + 1$ metre by n metre rectangle. The area of this combined figure is $2S_n = n(n + 1)$ square metres.

Another area-based approach

Consider once again the jagged triangular figure described in the previous proof. Another way to calculate the area of this figure is the following. First, draw a line from the bottom left corner of the first square on the left to the upper right corner of the top square on the right. The area of the triangle underneath this line is $n^2/2$. Above this line are n triangles of area 1/2 each. The total area S_n is then $n^2/2 + n/2 = n(n + 1)/2$.

Another box approach

Suppose we start with two unit squares lying horizontally next to each other. Call this the first configuration. This results in a 1×2 array of unit squares. The second configuration is obtained by adding a row of two squares above and another row below the first configuration to obtain a 3×2 group of unit squares. The third configuration is obtained by adding a column of three unit squares on the left and a column of three unit squares on the right to obtain a 3×4 array. Continuing this process of adding squares to the smaller side, one finds that the nth configuration is an $(n + 1) \times n$ array or an $n \times (n + 1)$ array of unit squares, depending on whether n is even or odd, respectively. In either case, there are $n(n + 1)$ unit squares in the nth configuration. Since the first configuration has two squares and there are $2(n + 1)$ additional squares each time we go from configuration n to configuration $n + 1$, it follows that the nth configuration has $2 + 4 + \ldots + 2n = 2(1 + 2 + \ldots + n) = 2S_n$ squares. Equating these two numbers gives the desired result $S_n = n(n + 1)/2$.

The next nine proofs are based on subjects familiar to most engineering students: linear algebra, elementary physics, probability theory, calculus, trigonometry, difference equations, and transform theory.

A matrix array approach

Consider an $(n + 1) \times (n + 1)$ matrix. There are of course $(n + 1)^2$ components in the matrix. A less direct way of counting the number of components is to separate the matrix into three parts, the diagonal part, the lower triangular part, and the upper triangular part. Clearly, there are $n + 1$ components along the diagonal. Examining the lower triangular part we see that there are $1 + 2 + \ldots + n = S_n$ components. This is also true for the upper triangular part. We therefore have $(n + 1)^2 = n + 1 + 2S_n$. Solving for S_n gives us the desired result. It is worthwhile to point out that this implies that there are generally $n(n + 1)/2$ independent elements in an $n \times n$ symmetric matrix. This is also true for the family of $n \times n$ upper (lower) triangular matrices. Similarly there are $n(n - 1)/2$ independent elements in an $n \times n$ skew-symmetric matrix.

A physics approach

Consider a see-saw whose sides are both n units long. Suppose that each side is labelled with n equally spaced marks and that two unit weights are placed on each mark. The see-saw will be balanced as the total torques $1(2) + 2(2) + \ldots + n(2) = 2S_n$ on each side are equal. Consider the right-hand side of the see-saw. The torque due to the two unit weights on the mark closest to the pivot is equal to the torque that would be exerted by a single weight at the second mark. Therefore replacing these two weights with an additional unit weight at the second mark does not change the total torque. On the right-hand side there are now no weights on the first mark, three weights on the second mark, and two weights on each of the remaining marks and the see-saw is still balanced. Replacing the three weights on the second mark with two additional weights on the third once again keeps the see-saw in balance. Next, replacing the four unit weights on the third mark with three additional unit weights at the fourth mark results in a right-hand side with no weights on the first three marks closest to the pivot, five unit weights on the fourth, and two unit weights on the remaining marks. As this does not change the total torque exerted on the right, the see-saw remains in balance. Continuing this process, we end up with a right-hand side having only $n + 1$ unit weights at its nth mark, resulting in a total torque of $n(n + 1)$ on the right, while the left-hand side still has its original configuration of two unit weights on each of its n unit-spaced marks resulting in a torque of $2S_n$ on the left. Since the see-saw is still balanced, we conclude that $2S_n = n(n + 1)$, proving the result.

A probability approach

The next proof is based on the observation that if a random variable **X** with a well-defined mean has a probability distribution that exhibits even symmetry about the value x_0, then the mean of **X** is x_0. Hence, a discrete random variable **X** that is uniformly distributed over the first n integers has a mean of $(n + 1)/2$. By definition, the mean is equal to $\mu_x = E(\mathbf{X}) = \Sigma_{k=1}^{n} x_k P(\mathbf{X} = x_k) = \Sigma_{k=1}^{n} k(1/n) = (1/n)\Sigma_{k=1}^{n} k$, proving the result.

A calculus-based approach

Consider the well-known geometric identity

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1} \tag{5}$$

which holds provided that $x \neq 1$. Differentiating gives

$$1 + 2x + \cdots + nx^{n-1} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}. \tag{6}$$

Letting x approach 1 and applying l'Hospital's rule twice gives the desired result. Some caution should be exercised when applying this type of approach due to the fact that $x - 1$ in the denominator approaches zero. The next proof uses another technique encountered in calculus.

Telescoping series

The idea of a telescoping series is based on the observation that for a given sequence P_n, we have that

$$P_n - P_0 = \sum_{k=1}^{n} (P_k - P_{k-1}) = \sum_{k=1}^{n} P_k - \sum_{k=0}^{n-1} P_k. \tag{7}$$

An example illustrating the use of a telescoping series is the following:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} = 1 - \frac{1}{n+1} = \frac{n}{n+1}. \tag{8}$$

To prove eqn (1) observe that

$$k = \frac{k(k+1)}{2} - \frac{k(k-1)}{2} = P_k - P_{k-1} \tag{9}$$

where $P_k = k(k + 1)/2$. Then

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n} (P_k - P_{k-1}) = P_n - P_0 = \frac{n(n+1)}{2} - 0 = \frac{n(n+1)}{2}. \tag{10}$$

A trigonometric identity

Consider the following well-known trigonometric identity:

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \sin\left(\frac{n\theta}{2}\right)\frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}. \tag{11}$$

Applying the small angle approximation to (11) gives

$$(1+2+\cdots+n)\theta \approx \frac{\frac{n\theta}{2}\left(\frac{n+1}{2}\right)\theta}{\frac{\theta}{2}} = \frac{n(n+1)}{2}\theta, \tag{12}$$

implying our result. This example illustrates how a student can check the accuracy of a formula by evaluating its limiting conditions. Not matching (1) would indicate that there is an error in the formula (11). In this way students can sometimes catch misremembered formulas. A more rigorous approach would have been to divide (11) by θ and rewrite the equation using the sinc function:

$$\sum_{k=1}^{n} k\,\mathrm{sinc}\,(k\theta) = \frac{n(n+1)}{2}\frac{\mathrm{sinc}\left(\frac{n}{2}\theta\right)\mathrm{sinc}\left(\frac{n+1}{2}\theta\right)}{\mathrm{sinc}\left(\frac{\theta}{2}\right)}, \tag{13}$$

where $\mathrm{sinc}(x) = \sin(x)/x$. Setting θ to zero gives the desired result. Another approach would have been to take a derivative of (11) and let θ approach zero; however, the resulting calculations are more involved than they were in the eleventh proof.

Difference equation approach

The quantity S_n is the solution to the linear difference equation

$$y_{n+1} - y_n = n+1, \quad y_1 = 1. \tag{14}$$

The general solution to a linear difference equation is given by its homogeneous and particular solutions. The homogeneous solution of (14) is $y_n^h = K$ while the particular solution has the form $y_n^p = an^2 + bn + c$. Substituting the latter expression into (14) one finds that $a = b = 1/2$. The general solution then is $y_n = n^2/2 + n/2 + K$. From the initial condition $y_1 = 1$ we find that $K = 0$ so $y_n = n(n + 1)/2$.

Z-transform

A systematic approach to solving linear difference equations like (14) is to use the Z-transform. This technique is commonly applied in digital control systems and digital signal processing. The procedure would be to take the Z-transform of both sides of (14), solve for the Z-transform $Y(z)$ of y_n, and use a standard Z-transform table to find that $y_n = n(n +)/2$. Since this is a fairly well-known technique the details are omitted.

Laplace transform

A time-domain expression for the figure described in the fifth proof is

$$f(t) = u(t) + u(t-1) + \cdots + u(t-n+1) - nu(t-n) \tag{15}$$

where $u(t)$ denotes the unit step function. The Laplace transform for this expressions is

$$F(s) = \frac{1 + e^{-s} + \cdots + e^{-(n-1)s} - ne^{-ns}}{s}. \tag{16}$$

By the final value theorem we have that

$$\begin{aligned} \int_0^\infty f(t)\,\mathrm{dt} &= \lim_{s\to 0} F(s) \\ &= \lim_{s\to 0} \frac{1 + e^{-s} + \cdots + e^{-(n-1)s} - ne^{-ns}}{s} \\ &= -\big(1 + 2 + \cdots + (n-1)\big) + n^2 \\ &= -S_n + n + n^2 \end{aligned} \tag{17}$$

where the third equality follows from l'Hospital's rule. Since (17) is equal to S_n, we conclude that $S_n = n(n + 1)/2$.

In addition to proving (1), the next four proofs have the advantage of giving more general results.

A modern algebra approach

Modulo arithmetic has a number of electrical engineering applications including coding theory and the generation of random numbers. It can also be used to prove (1). First, we show that for positive odd integers l, the sum $S_{l,n} = \Sigma_{k=1}^{n} k^l$ is divisible by $n(n + 1)/2$. Then we show that for $l = 1$ this ratio is unity. Now for odd l we have

$$\begin{aligned} 2S_{l,n} &\equiv 1^l + 2^l + \cdots + n^l + 1^l + 2^l + \cdots + n^l \pmod{n+1} \\ &\equiv 1^l + 2^l + \cdots + n^l + (-n)^l + [-(n-1)]^l + \cdots + (-1)^l \pmod{n+1} \\ &\equiv 1^l + 2^l + \cdots + n^l - n^l - (n-1)^l - \cdots - 1^l \pmod{n+1} \\ &\equiv 0 \pmod{n+1} \end{aligned} \tag{18}$$

where we have used the fact that $-k \equiv n + 1 - k \pmod{n + 1}$ and the fact that for odd l, $(-k)^l = -k^l$. We therefore have that $n + 1$ divides $2S_{l,n} = 2(1^l + 2^l + \ldots + n^l)$. Similarly, n divides $2[1^l + 2^l + \ldots + (n - 1)^l]$, which follows by replacing n with $n - 1$ in the above argument. As n also divides $2n^l$ we have that n divides $2(1^l + 2^l + \ldots + n^l) = 2S_{l,n}$. Because n and $n + 1$ are relatively prime, it further follows that $n(n + 1)$ divides $2S_{l,n}$. Furthermore, since the product of any two consecutive integers is even, 2 divides $n(n + 1)$ and $2S_{l,n}$. We therefore conclude that $n(n + 1)/2$ divides $S_{l,n}$ so that $S_{l,n} = a_n n(n + 1)/2$ where a_n is an integer for each n. For the case when $l = 1$, a_n satisfies the difference equation $(n + 2)a_{n+1} - na_n = 2$ with $a_1 = 1$. Since this difference equation has the unique solution $a_n = 1$, it follows that $S_n = n(n + 1)/2$ and that S_n divides $\Sigma_{k=1}^{n} k^l$ for every positive odd integer l. We will see in the next proof that this last statement is generally not true when l is even.

A combinatorial proof

Students who are familiar with combinatorics may notice that the right-hand side of eqn (1) is $\binom{n+1}{2}$. This suggests that there may be a combinatorial proof of this identity and indeed there is. Note that the number of ways to choose two distinct numbers from the set $\{1, 2, \ldots, n + 1\}$ is

$$\binom{n+1}{2} = \sum_{k=1}^{n} N_k = \sum_{k=1}^{n} k \tag{19}$$

where $N_k = k$ is the number of ways one can choose two distinct numbers from $\{1, 2, \ldots, n + 1\}$ with the larger number being $k + 1$. This proves (1). More generally, we have that

$$\binom{n+m}{m+1} = \sum_{k=1}^{n} N_k = \sum_{k=1}^{n} \binom{k+m-1}{m} \tag{20}$$

where N_k now denotes the number of ways one can choose $m + 1$ distinct numbers from $\{1, 2, \ldots, n + m\}$ with the largest of the $m + 1$ numbers being $k + m$. Equation (20) can also be derived using a telescoping series based on a rearrangement of Pascal's identity.

Since k^l is a linear combination of terms like $\binom{k+n_q}{q}$ with $q = 0, 1, 2, \ldots, l$, we can use the identity (20) to obtain solutions to $\Sigma_{k=1}^{n} k^l$. This results in particularly simple formulas for the cases when $l = 2$ and 3. Noting that

$$k^2 = \frac{k(k+1) + k(k-1)}{2} = \binom{k+1}{2} + \binom{k}{2} \tag{21}$$

and

$$k^3 = (k+1)k(k-1) + k = 3!\binom{k+1}{3} + k, \tag{22}$$

we have that

$$\sum_{k=1}^{n} k^2 = \binom{n+2}{3} + \binom{n+1}{3} = \frac{n(n+1)(2n+1)}{6} \tag{23}$$

and

$$\sum_{k=1}^{n} k^3 = 3!\binom{n+2}{4} + \binom{n+1}{2} = \binom{n+1}{2}^2 = \left(\sum_{k=1}^{n} k\right)^2 \tag{24}$$

Note that $\Sigma_{k=1}^{n} k^3$ is divisible by $S_n = n(n+1)/2$ as was shown in the 17th proof. However, the integer $n(n+1)/2$ does not generally divide $\Sigma_{k=1}^{n} k^2$ as $(2n+1)/3$ is typically not an integer.

Our final two proofs give systematic approaches for determining closed form expressions for the more general series $\Sigma_{k=1}^{n} k^l$ where l is any fixed positive integer.

Polynomial interpolation

Now that we are convinced that S_n is a second-order polynomial in n, we can use interpolation to find this polynomial. Polynomial interpolation is based on the well-known fact that any m-th order polynomial $p(x) = a_m x^m + \ldots + a_1 x + a_0$ is uniquely determined by $m + 1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_m, y_m)$ with $y_i = p(x_i)$ provided of course that $x_0, x_1, \ldots, x_m$ are distinct. A common polynomial interpolation technique is Lagrange interpolation which is given by Ref. [4] as

$$p(x) = \sum_{k=0}^{m} y_k p_k(x) \tag{25}$$

where the Lagrange interpolation polynomials for $x_0, x_1, \ldots, x_m$ are defined as

$$p_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{m} \frac{x - x_j}{x_k - x_j}, \quad k = 0, 1, \ldots, m. \tag{26}$$

The symbol $\prod_{\substack{j=0 \\ j \neq k}}^{m} z_j$ denotes the product of all the terms $z_0, z_1, \ldots, z_m$ except z_k. The interpolation polynomials $p_k(x)$ satisfy $p_k(x_j) = \delta_{jk}$ where $\delta_{jk} = 1$ if $j = k$ and 0 if $j \neq k$ so that $p(x_i) = y_i$ as desired.

Since S_n is a second order polynomial in n, we can apply Lagrange interpolation using $S_1 = 1$, $S_2 = 3$, and $S_3 = 6$ to obtain

$$S_n = \frac{(n-2)(n-3)}{(1-2)(1-3)} + 3\frac{(n-1)(n-3)}{(2-1)(2-3)} + 6\frac{(n-1)(n-2)}{(3-1)(3-2)}. \tag{27}$$

Another polynomial interpolation technique is to use the Vandermonde matrix.[4] In this method, the set of equations

$$y_i = a_0 + a_1 x_i + \ldots + a_m x_i^m, \quad i = 0, 1, \ldots, m \tag{28}$$

are written in matrix form

$$\mathbf{y} = V(x_0, x_1, \ldots, x_m)\mathbf{x} \tag{29}$$

where $\mathbf{x} = [x_0 \quad x_1 \quad \cdots \quad x_m]^T$ and $\mathbf{y} = [y_0 \quad y_1 \quad \cdots \quad y_m]^T$ are $(m + 1)$-vectors and

$$V(x_0, x_1, \ldots, x_m) = \begin{bmatrix} 1 & x_0 & \cdots & x_0^m \\ 1 & x_1 & \cdots & x_1^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \cdots & x_m^m \end{bmatrix} \tag{30}$$

is the $(m + 1) \times (m + 1)$ Vandermonde matrix. The Vandermonde matrix appears in a number of applications including signal processing and has several interesting properties. In particular, its determinant is given by

$$\det\left[V(x_0, x_1, \ldots, x_m)\right] = \prod_{i<j} (x_j - x_i) \tag{31}$$

so that $V(x_0, x_1, \ldots, x_m)$ is invertible if and only if $x_0, x_1, \ldots, x_m$ are distinct. In this case the coefficients of $p(x)$ are uniquely determined by (29) as $\mathbf{a} = V^{-1}\mathbf{y}$. For the second order polynomial S_n we solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \tag{32}$$

to find the coefficients of $S_n = a_2n^2 + a_1n + a_0$. Since $\Sigma_{k=1}^{n} k^l$ is an $(l + 1)$th order polynomial, we can use polynomial interpolation to determine expressions for it in essentially the same way.

Euler's summation formula

Euler's summation formula is a method for obtaining approximations to a finite sum using an integral. Before giving Euler's summation formula, we need to discuss Bernoulli numbers and polynomials. Bernoulli numbers first appeared in connection with Jakob Bernoulli's study of the sum of powers of consecutive integers $\Sigma_{k=1}^{n} k^l$. The Bernoulli numbers B_k are defined by the generating function

$$\frac{z}{e^z - 1} = B_0 + B_1 z + \frac{B_2 z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{B_k z^k}{k!}. \tag{33}$$

For example, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, and $B_4 = -1/30$. The remaining odd Bernoulli numbers are zero: $B_3 = B_5 = B_7 = \cdots = 0$. The Bernoulli polynomial is defined by

$$B_m(x) = \sum_{k=0}^{m} \binom{m}{k} B_k x^{m-k}. \tag{34}$$

We can now state Euler's summation formula:

$$\sum_{1 \le k < n} f(k) = \int_1^n f(x)\,dx + \sum_{k=1}^{m} \frac{B_k}{k!}\left[f^{(k-1)}(n) - f^{(k-1)}(1)\right] + R_{mn} \tag{35}$$

where

$$R_{mn} = \frac{(-1)^{m+1}}{m!}\int_1^n B_m(x-\lfloor x\rfloor)f^{(m)}(x)\,dx \tag{36}$$

and where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. A derivation of this result can be found in Ref. [2].

Using (35) with $f(x) = x$ and $m = 2$, we obtain

$$\sum_{k=1}^{n} k = \int_1^{n+1} x\,dx - \frac{1}{2}[n+1-1] = \frac{n(n+1)}{2}. \tag{37}$$

Note that the calculation of (36) was trivial because the term R_{mn} is zero for $m = 2$ as $f^{(2)}(k) = 0$ in this case. One can readily calculate $\Sigma_{k=1}^{n} k^l$ for any positive integer l using this method with $f(x) = x^l$. By taking $m = l + 1$, the term R_{mn} is automatically zero as $f^{(m)}(x) = 0$.

Although there are other proofs of (1), this modest collection should convince students that there are many feasible approaches to a problem. After studying the proofs in this section, students should see how many ways they can prove the identity

$$1+3+\cdots+(2n-1) = n^2, \tag{38}$$

i.e., that the sum of the first n odd integers is equal to n^2. Indeed, many of the approaches above can be easily modified to prove (38). After proving (38), students should prove the more general arithmetic series formula: $a + (a + b) + \ldots + (a + nb) = (n + 1)(2a + nb)/2$. Many other types of results can also be proven in this way.

A warning about false proofs

Before concluding this article, we point out one of the hazards facing a beginning student: the false proof. An example of a type of false proof is a proof that uses circular logic. For example, one cannot use trigonometry to prove the Pythagorean Theorem since the theorem is inherently used in the definition of trigonometric functions.[3] There are many other examples of false proofs but we shall only give two more. They result, respectively, from ignoring the conditions under which a statement is true and making assumptions about how operations behave based on their behaviour under more familiar but less general conditions.

Consider the following infinite series:

$$\begin{aligned}\sum_{k=-\infty}^{\infty} x^k &= \sum_{k=1}^{\infty} x^k + \sum_{k=0}^{\infty} x^{-k} = x\sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty}\frac{1}{x^k} \\ &= \frac{x}{1-x} + \frac{1}{1-(1/x)} = \frac{x}{1-x} + \frac{x}{x-1} \\ &= \frac{x}{1-x} - \frac{x}{1-x} = 0.\end{aligned} \tag{39}$$

Obviously there is something wrong, for if x is any positive number, then every power of x is also positive, so how can we get a zero here? This type of error was not uncommon in the 18th century when mathematicians were just starting to understand the concept of a region of convergence. The region of convergence for the first term $\Sigma_{k=1}^{\infty} x^k$ is $|x| < 1$ and for the second term $\Sigma_{k=0}^{\infty} x^{-k}$ it is $|x| > 1$, but the region of convergence for the complete sum is the empty set. So this series diverges for every value of x. The application of the geometric series formula without checking the hypothesis about the region of convergence resulted in an incorrect result.

A lack of understanding of complex numbers has also resulted in notable errors in the work of some very prominent mathematicians of the past. This is not surprising as a complete understanding and acceptance of complex numbers took a surprisingly long time. Although complex numbers were introduced in the 16th century to aid in the solution of polynomial equations, viz., the cubic polynomial, even the most respected mathematicians were making errors like $\sqrt{-2}\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}$ in the 18th century. This error results from a false assumption that the square root operator works in the same way for negative numbers as it does for positive numbers. Of course writing $\sqrt{-2} = j\sqrt{2}$ and $\sqrt{-3} = j\sqrt{3}$ gives the correct result $-\sqrt{6}$.

Conclusions

In this article, a case study detailing a number of proofs of a simple well-known identity was given. By using techniques that are within the repertoire of most students, these proofs should serve as a motivation for undergraduate engineering students to come up with their own problems so they can practise this important skill. Students who become proficient at proving results will find that they have a better understanding of course material, are more likely to catch errors, have an increased ability to develop creative solutions, and are better prepared for graduate school. Lastly, a cautionary warning was given that one should be careful to avoid mistakes that lead to false proofs.

References

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