On the use of auxiliary vector potentials as a common tool for studying electromagnetic problems at undergraduate level

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Abstract Applications of Maxwell's equations to electromagnetic problems can be divided into two large groups: one dealing with radiation and scattering and the other with propagation. In this paper it is shown how both kinds of problem can be managed by means of the auxiliary vector potentials $\vec{A}$ and $\vec{F}$.

Keywords auxiliary potentials; electromagnetic theory; Maxwell's equations; TEM modes

The electromagnetic behaviour of physical phenomena is described by Maxwell's equations. An understanding and application of these equations is fundamental to electrodynamics courses in physics and engineering.

It is common practice in the analysis of electromagnetic problems to use auxiliary vector potentials as an aid to obtaining the electromagnetic field vectors. A family of vector potential functions is formed by $\vec{A}$, the magnetic vector potential, and $\vec{F}$, the electric vector potential.[1,2] They are used in a great variety of electromagnetic problems such as radiation from linear and aperture antennas[3,4] and propagation problems in empty or partially filled rectangular and circular waveguides.[1] Another family is the pair of Hertz vector potentials $\vec{\Pi}_e$ and $\vec{\Pi}_h$,[5,6] where the vector potential $\vec{\Pi}_e$ is analogous to $\vec{A}$ and $\vec{\Pi}_h$ is analogous to $\vec{F}$. They are related by a proportionality constant that is a function of the frequency and the electromagnetic parameters of the medium. In this paper we will use $\vec{A}$ and $\vec{F}$, and these auxiliary potential functions may or may not represent clearly definable physical entities. For most engineers the auxiliary potentials are just useful mathematical functions from which the electromagnetic field can be derived.

Electromagnetic problems involving radiation and scattering are handled by means of the auxiliary vector potentials, either $\vec{A}$ and $\vec{F}$ or the Hertz vector potentials. For propagation problems there are two alternatives: in one of them, Maxwell's equations are particularised for each specific type of mode whereas, in the other, the auxiliary potentials are still used, particularising them to the source-free case. However, this latter alternative involves a difficulty in the procedure: when the fields of the TEM modes are obtained as a function of the auxiliary vector potentials, there is no report in the literature showing how these electric and magnetic fields can be derived from a scalar potential Φ, which is an essential step in order to introduce the unique voltage and current waves associated with a TEM mode in an ideal transmission line[7,8] and, subsequently, distributed circuit theory. Figure 1 summarizes all these comments.

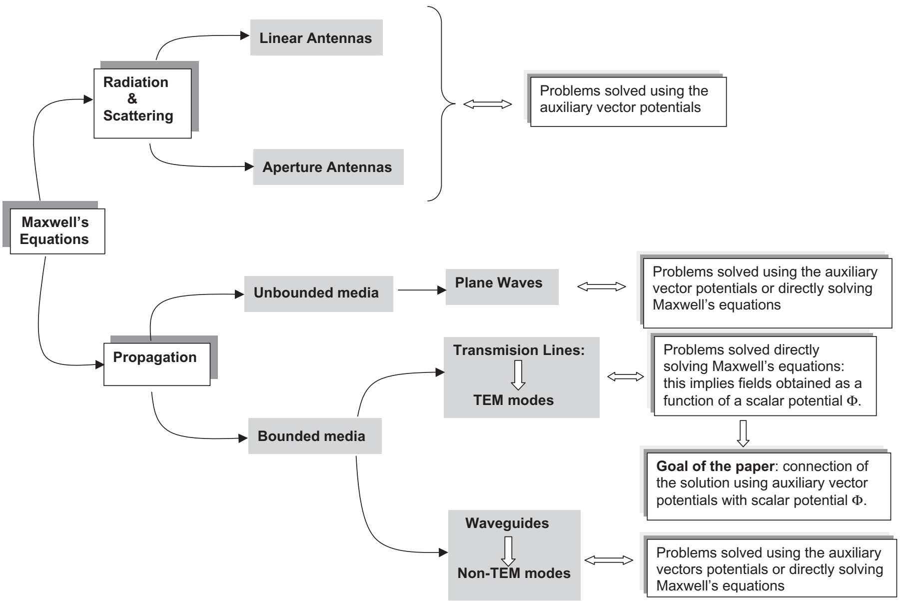

Fig. 1 *Block diagram showing a breakdown of the study of electromagnetic waves by means of Maxwell's equations. Also shown is the generic kind of problem analysed in an undergraduate-level electromagnetics course in engineering and physics and the type of field solutions corresponding to each one. The tools used in the literature for solving the different problems are also shown.*

In other words, we have noticed that, in the literature available related to the propagation of electromagnetic waves, no connection is made between the process of obtaining the TEM modes from the auxiliary vector potentials (field theory) and the unique voltage and current waves associated with each mode (circuit theory), in the same manner as if this task is made by obtaining the TEM modes directly from Maxwell's equations, as can be seen in Refs [7] and [8]. For example, Balanis[1] shows how to obtain the fields of a TEM mode, but not how these fields, which are transverse to the direction of propagation, can be obtained by means of a scalar potential function, which is a solution of the Laplace equation. This step provides the key for introducing the voltage and current waves.[7,8] On the other hand, Collin[7] or Pozar[8] show the development of this step but with the electromagnetic field obtained directly from Maxwell's equations.

Now, we can return to Fig. 1 in order to understand the process followed in the study of electromagnetism in the 4th year physics electronics course at the University of Cantabria and, consequently, to explain the goal of this paper. Figure 1 summarises the following: the course begins with Maxwell's equations and then continues with radiation of linear antennas (aperture antennas are also mentioned, although they are studied in the next course), plane waves, transmission lines (from the point of view of distributed circuit parameters and field theory) and waveguides.

Radiation from linear antennas is analysed using the vector potential $\vec{A}$. Plane waves can then be studied directly from Maxwell's equations or from the vector potentials. The study of transmission lines is not carried out using the vector potentials, because of the problem mentioned above; instead, the TEM modes are obtained directly from Maxwell's equations (e.g. Collin, Pozar[7,8]). Finally, the analysis of TE, TM, LSE and LSM modes in empty and partially filled waveguides is again made by means of the vector potentials. As can be seen, there is a 'discontinuity' when the study of transmission lines is reached, which is difficult to explain. So, we have considered the following question: how can the above topics be explained with the same tool, i.e., the vector potentials? The process is shown in this paper.

TEM modes from $\vec{A}$ and $\vec{F}$ potentials

Since $\nabla\cdot\vec{B} = 0$, it is possible to define a vector, which obeys the identity

$$\nabla\cdot\nabla\times\vec{A} = 0 \tag{1}$$

where $\vec{A}$ is defined as the magnetic vector potential. Therefore,

$$\vec{B}_A = \mu\vec{H}_A = \nabla\times\vec{A} \tag{2}$$

where the A subscript indicates that the field is due to the magnetic field vector. In a similar manner, in a source free region, $\nabla\cdot\vec{D} = 0$ and an electric vector potential may be defined through the following relationship

$$\nabla\cdot\left(-\nabla\times\vec{F}\right) = 0 \tag{3}$$

therefore

$$\vec{D}_F = \varepsilon\vec{E}_F = -\nabla\times\vec{F} \tag{4}$$

where the subscript F indicates that the fields are due to the electrical vector potential $\vec{F}$. The above definitions imply that the vector potentials can be considered as the sources of the electric and magnetic fields instead of the electric and magnetic charges.

Using the above definitions of $\vec{A}$ and $\vec{F}$, and following the formulation presented in Ref. [1] for sinusoidal varying fields, we are able to express the electric and magnetic field as

$$E = \vec{E}_A + \vec{E}_F = -j\omega\vec{A} - j\frac{1}{\omega\mu\varepsilon}\vec{\nabla}\left(\nabla\cdot\vec{A}\right) - \frac{1}{\varepsilon}\nabla\times\vec{F} \tag{5a}$$

$$\vec{H} = \vec{H}_A + \vec{H}_F = \frac{1}{\mu}\nabla\times\vec{A} - j\omega\vec{F} - \frac{j}{\omega\mu\varepsilon}\vec{\nabla}\left(\nabla\cdot\vec{F}\right) \tag{5b}$$

In the following discussion, we will use rectangular coordinates; however, the same could be done in cylindrical coordinates.

The vector potentials $\vec{A}$ and $\vec{F}$ can be written in rectangular coordinates as

$$\vec{A}(x,y,z) = \vec{a}_x A_x(x,y,z) + \vec{a}_y A_y(x,y,z) + \vec{a}_z A_y(x,y,z) \tag{6a}$$

$$\vec{F}(x,y,z) = \vec{a}_x \vec{F}_x(x,y,z) + \vec{a}_y \vec{F}_y(x,y,z) + \vec{a}_z \vec{F}_y(x,y,z) \tag{6b}$$

where both satisfy the source-free Helmholtz equation

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0 \quad \text{and} \quad \nabla^2 \vec{F} + k^2 \vec{F} = 0 \tag{7}$$

where k is the wave number in the medium $k = \omega\sqrt{\mu\varepsilon}$. Expanding eqns (5a) and (5b) into their components, it is possible to obtain the electromagnetic field as a function of the vector potential components

$$\begin{aligned}\vec{E} = \vec{a}_x &\left\{ -j\omega A_x - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}\right) - \frac{1}{\varepsilon}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\right\} \\ + \vec{a}_y &\left\{ -j\omega A_y - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x \partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y \partial z}\right) - \frac{1}{\varepsilon}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\right\} \\ + \vec{a}_z &\left\{ -j\omega A_z - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial y \partial z} + \frac{\partial^2 A_z}{\partial z^2}\right) - \frac{1}{\varepsilon}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\right\}\end{aligned} \tag{8}$$

$$\begin{aligned}\vec{H} = \vec{a}_x &\left\{ -j\omega F_x - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial x \partial y} + \frac{\partial^2 F_z}{\partial x \partial z}\right) - \frac{1}{\mu}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right\} \\ + \vec{a}_y &\left\{ -j\omega F_y - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x \partial y} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial y \partial z}\right) - \frac{1}{\mu}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right\} \\ + \vec{a}_z &\left\{ -j\omega F_z - j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x \partial z} + \frac{\partial^2 F_y}{\partial y \partial z} + \frac{\partial^2 F_z}{\partial z^2}\right) - \frac{1}{\mu}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right\}\end{aligned} \tag{9}$$

With the above equations it is relatively simple to obtain solutions for the TE, TM and TEM modes in any direction. In this work we will focus only on the TEM modes in the z direction. The procedure is similar for the x and y directions.

According to Ref. [1], the TEM modes in the z direction can be obtained in any of the three following ways

1. $A_x = A_y = F_x = F_y = 0; \quad A_z \neq 0, \quad F_z \neq 0, \quad \dfrac{\partial}{\partial x} \neq 0, \quad \dfrac{\partial}{\partial y} \neq 0$ (10a)

2. $A_x = A_y = A_z = F_x = F_y = 0; \quad F_z \neq 0, \quad \dfrac{\partial}{\partial x} \neq 0, \quad \dfrac{\partial}{\partial y} \neq 0$ (10b)

3. $A_x = A_y = F_x = F_y = F_z = 0; \quad A_z \neq 0, \quad \dfrac{\partial}{\partial x} \neq 0, \quad \dfrac{\partial}{\partial y} \neq 0$ (10c)

Any of these will result in a TEM mode in the z direction, i.e., a mode in which $E_z = 0$ and $H_z = 0$.

We will consider the more general case, given by eqn (10a), for which it is easy to show that both A_z and F_z must satisfy the equation

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) A_z, \quad F_z = 0 \tag{11}$$

for which, considering propagation only along the z positive direction and time harmonic variations of the form $e^{j\omega t}$, the solution becomes

$$A_z(x, y, z) = A_z^+(x, y)e^{-jkz}; \quad F_z(x, y, z) = F_z^+(x, y)e^{-jkz} \tag{12}$$

where $A_z^+(x,y)$ $F_z^+(x,y)$ are the amplitudes of each potential. The above equations show an essential property of a TEM mode: the phase constant is the wave number in the medium and, therefore, there is no cut-off frequency.

Now, using the set of equations (8) and (9), it can be easily shown that

$$E_x = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial x} - \frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial y}\right)e^{-jkz}; \quad E_y = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_z^+}{\partial x} + \frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial y}\right)e^{-jkz} \tag{13a}$$

$$H_x = -\sqrt{\frac{\varepsilon}{\mu}}E_y = -Y_\omega E_y; \quad H_y = \sqrt{\frac{\varepsilon}{\mu}}E_x = Y_\omega E_y \tag{13b}$$

where Y_ω is the wave admittance of a TEM mode, which coincides with the intrinsic admittance of the medium. Hence, the relation between the electric and magnetic field is

$$\vec{H} = Y_\omega \vec{a}_z \times \vec{E} \tag{14}$$

Connection with voltage and current waves

So far, we have shown how to obtain the equations of a TEM mode in the usual way, for example, using the same approach as Balanis.[1] However, our final objective is to prove that the electric field (and therefore the magnetic field) of a TEM mode can be obtained from a scalar potential that satisfies the Laplace equation. This will serve to show that the fields, and the electric field in particular, vary spatially according to the transverse coordinates, in the same way as an electrostatic field. Therefore, with respect to such coordinates they are conservative fields and we can find a unique set of voltage and current waves from which circuit models of transmission lines can be obtained. As a consequence, it is possible to establish a relationship with the circuit theory of distributed parameters useful for the analysis of transmission lines.

To prove this, first it will be shown that the transverse rotational of the electric field is zero, for which first, we write the electric field as

$$\vec{E}(x, y, z) = \vec{e}(x, y)e^{-jkz} = (\vec{a}_x e_x(x, y) + \vec{a}_y e_y(x, y))e^{-jkz} \tag{15}$$

where e_x and e_y can be obtained from eqn (13a) simply by inspection. The transverse rotational (i.e. rotational respect to the transverse coordinates as it is used in Ref. [7]) of $\vec{e}(x,y)$ is therefore

$$\vec{\nabla} \times \vec{e} = \vec{a}_z\left(\frac{\partial e_y}{\partial x} - \frac{\partial e_x}{\partial y}\right) = \vec{a}_z\frac{1}{\varepsilon}\left(\frac{\partial^2 F_z^+}{\partial x^2} + \frac{\partial^2 F_z^+}{\partial y^2}\right) = \vec{a}_z\frac{1}{\varepsilon}\nabla_t^2 F_z^+(x, y) \tag{16}$$

Since the vector potential $\vec{F}$ satisfies the Helmholtz equation, for variations in z of the form e^{-jkz} we have

$$\nabla^2 F_z + k^2 F_z = 0 \Rightarrow (\nabla_t^2 - k^2)F_z + k^2 F_z = 0 \Rightarrow \nabla_t^2 F_z = 0 \Rightarrow \nabla^2 F_z^+ = 0 \quad (17)$$

hence, by means of eqn (16), it is evident that

$$\vec{\nabla}_t \times \vec{e}(x, y) = 0 \quad (18)$$

Also, the transverse divergence of $\vec{e}(x,y)$ can be easily obtained. Since a source-free medium is assumed, the divergence of $\vec{E}$ is zero; moreover, since $E_z = 0$ we can write

$$\vec{\nabla}_t \cdot \vec{e}(x, y) = 0 \quad (19)$$

Using eqn (18) it is now possible to write

$$\vec{e}(x, y) = -\vec{\nabla}_t \Phi(x, y) \Rightarrow \vec{E}(x, y, z) = -\vec{\nabla}_t \Phi(x, y) e^{-jkz} \quad (20)$$

where $\Phi\ (x,y)$ is the potential function that satisfies the Laplace equation. In order to see this, it is sufficient to take the transverse divergence of $\vec{e}(x,y)$, which, as stated above, is equal to zero

$$\vec{\nabla}_t \cdot \vec{e}(x, y) = -\vec{\nabla}_t \cdot \vec{\nabla}_t \Phi(x, y) = 0 \Rightarrow \nabla^2 \Phi(x, y) = 0 \quad (21)$$

From eqn (20), it is straightforward to obtain the equation for a unique voltage wave associated with the electric field[7]

$$V = V_0 e^{-jkz} \quad (22)$$

where V_0 is the line integral of the potential gradient $\nabla_t \Phi\ (x,y)$ between two points s_1 and s_2 on the conductors s_1 and s_2[7]

$$V_0 = \int_{s_1}^{s_2} \vec{e} \cdot d\vec{l} = \int_{s_1}^{s_2} -\vec{\nabla}_t \Phi \cdot d\vec{l} = -(\Phi(s_2) - \Phi(s_1)) \quad (23)$$

Conclusions

In this paper we have described a simple formulation that allows us to connect the method of obtaining the fields corresponding to a TEM mode by means of the auxiliary vector potentials $\vec{A}$ and $\vec{F}$ with the method of obtaining the unique voltage and current waves associated with it. For this purpose, we have shown that these fields can be expressed through a scalar potential that satisfies the Laplace equation. This closes the existing gap for those lecturers who choose, not to use Maxwell's equations directly when applying Maxwell's electromagnetic theory, but auxiliary vector potentials such as $\vec{A}$ and $\vec{F}$.

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