Plane wave visualisation on a dielectric-dielectric interface

Mohammed H. Bataineh
Communications Engineering Department, Hijjawi Faculty for Engineering Technology, Yarmouk University, Irbid, Jordan
E-mail: mohbat@yu.edu.jo

Abstract In this paper, the behaviour of an arbitrarily polarised plane wave on a dielectric-dielectric interface is investigated. By imposing the appropriate boundary conditions the Fresnel transmission and reflection coefficients are derived. Using these coefficients various three dimensional as well as contour plots are produced using a MATLAB code in order to illustrate the refraction and the reflection usually encountered at an interface. Snell's law and the Brewster angle are discussed within the context of the analysis.

Keywords boundary conditions; electromagnetic plane wave; MATLAB; Snell's law

When an arbitrarily polarised electromagnetic plane wave impinges upon an interface it may be decomposed with respect to the plane of incidence to a perpendicularly and horizontally polarised wave. In the perpendicular polarisation case, the electric field is transverse to the plane of incidence; hence we refer to this polarisation as *t*ransverse *e*lectric, or TE-polarisation. On the other hand, when the magnetic field is transverse to the plane of incidence we refer to this polarisation as *t*ransverse *m*agnetic, or TM-polarisation.[1]

In this paper, a MATLAB code is given to clarify the refraction and the reflection of a vector field at a planar interface separating two dielectric media. Three-dimensional and contour plots are generated for a variety of constitutive parameters and angle of incidence. The polarisation type and the angle of incidence affect the reflection and the transmission coefficients formulae. The well-known Snell's law is given by 'phase-matching' the fields at the interface. Another interesting limiting incident angle is the polarisation or the Brewster angle at which the incident power is totally transmitted. Fortunately this occurs only for TM-polarisation.[2] Therefore, the interface may be regarded as a polariser and hence the name attributed to this angle is the polarisation angle. In the second section, formulae of the reflection and transmission coefficients for each polarisation are given in terms of the constitutive parameters and the incident wave angle. The third section gives some specific examples especially for critical cases. Finally, the conclusions are summarised.

Analysis

In this section, the various Fresnel coefficients will be derived taking the xz-plane as the incident plane as shown in Fig. 1. The direction of propagation, which is along the wavenumber vector $\vec{k}$, and the normal vector on the interface span the plane of incidence. For oblique incidence, the electric field may lie on this plane or perpen-

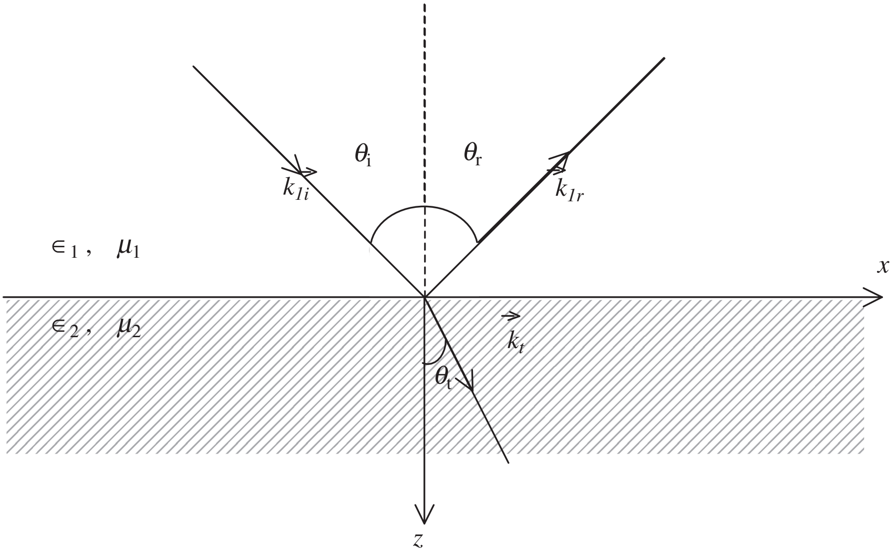

Fig. 1 *Planar interface between two media.*

dicular to it. The latter is referred to as TE-polarisation while the former is called TM-polarisation.

Perpendicular polarisation

In the perpendicular polarisation (TE) case, the incident electric field $\vec{E}_i$ is directed out of the page in the $\hat{a}_y$ direction. Using the complex phasor notation, we may write

$$\vec{E}_i = \hat{a}_y E_{0i} e^{-j\vec{k}_{1i}\cdot\vec{r}} \tag{1}$$

The incident wavenumber in the upper half-space $\vec{k}_{1i}$ is given by

$$\vec{k}_{1i} = k_1(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i) \tag{2}$$

Here, $k_1 = \omega\sqrt{\mu_1\varepsilon_1}$ is the magnitude of the wavenumber for the upper half-space. The corresponding magnetic field may be written as

$$\begin{aligned}\vec{H}_i &= \frac{\vec{k}_{1i} \times \vec{E}_i}{\omega\mu_1} \\ &= \frac{E_{0i}}{Z_1}(-\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_i)e^{-j\vec{k}_{1i}\cdot\vec{r}}\end{aligned} \tag{3}$$

where $Z_1 = \sqrt{\mu_1/\varepsilon_1}$ is the intrinsic wave impedance for the first medium. Upon reaching the boundary at the $z = 0$ plane, part of the wave will be reflected and part will be transmitted or refracted. The corresponding expressions for these field quantities are given by

$$\vec{E}_r = \hat{a}_y E_{0r} e^{-j\vec{k}_{1r}\cdot\vec{r}} \tag{4}$$

and

$$\vec{H}_r = \frac{\vec{k}_{1r} \times \vec{E}_r}{\omega\mu_1}$$
$$= \frac{E_{0r}}{Z_1}(\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r)e^{-j\vec{k}_{1r}\cdot\vec{r}} \tag{5}$$

Here,

$$\vec{k}_{1r} = k_1(\hat{a}_x \sin\theta_r - \hat{a}_z \cos\theta_r) \tag{6}$$

At this point we can write the total field quantities in the upper half-space as

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r \tag{7}$$

and

$$\vec{H}_1 = \vec{H}_i + \vec{H}_r \tag{8}$$

The transmitted field quantities may be written as

$$\vec{E}_t = \hat{a}_y E_{0t} e^{-j\vec{k}_t\cdot\vec{r}} \tag{9}$$

Here,

$$\vec{k}_t = k_2(\hat{a}_x \sin\theta_t + \hat{a}_z \cos\theta_t) \tag{10}$$

with $k_2 = \omega\sqrt{\mu_2\varepsilon_2}$. The magnetic field intensity is given by

$$\vec{H}_t = \frac{\vec{k}_t \times \vec{E}_t}{\omega\mu_2}$$
$$= \frac{E_{0t}}{Z_2}(-\hat{a}_x \cos\theta_t + \hat{a}_z \sin\theta_t)e^{-j\vec{k}_t\cdot\vec{r}} \tag{11}$$

where $Z_2 = \sqrt{\mu_2/\varepsilon_2}$ is the intrinsic wave impedance for the second medium. Imposing the boundary conditions at the interface $z = 0$ requires the continuity of the tangential components of the electric and the magnetic field intensities. The first of these conditions leads to

$$\vec{E}_1\Big|_{z=0} \cdot \hat{a}_y = \vec{E}_t\Big|_{z=0} \cdot \hat{a}_y \tag{12}$$

or,

$$E_{0i}e^{-jk_1 \sin\theta_i x} + E_{0r}e^{-jk_1 \sin\theta_r x} = E_{0t}e^{-jk_2 \sin\theta_t x} \tag{13}$$

and this should be valid irrespective of the value of x, i.e., the following should be satisfied

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t \tag{14}$$

This condition is commonly referred to as 'phase matching' and leads to the following Snell's laws of reflection and refraction

$$\theta_i = \theta_r \tag{15}$$

$$k_1 \sin\theta_i = k_2 \sin\theta_t \tag{16}$$

Upon using eqn (14), eqn (13) may be rewritten as

$$1 + \frac{E_{0r}}{E_{0i}} = \frac{E_{0t}}{E_{0i}} \tag{17}$$

The appropriate ratios of the electric field denote the Fresnel reflection and transmission coefficients as defined respectively by

$$\Gamma_\perp = \frac{E_{0r}}{E_{0i}} \tag{18}$$

$$T_\perp = \frac{E_{0t}}{E_{0i}} \tag{19}$$

The symbol $\perp$ stands for 'perpendicular'. In order to write an explicit expression for these two coefficients, the other boundary condition for the magnetic field intensity is applied yielding

$$\vec{H}_1\Big|_{z=0} \cdot \hat{a}_x = \vec{H}_t\Big|_{z=0} \cdot \hat{a}_x \tag{20}$$

or,

$$\frac{\cos\theta_i}{Z_1}(1-\Gamma_\perp) = \frac{\cos\theta_t}{Z_2} T_\perp \tag{21}$$

where substitutions from eqn (14) and eqn (15) have been made to write eqn (21). Solving eqn (17) with (21) to arrive at

$$\Gamma_\perp = \frac{Z_2 \cos\theta_i - Z_1 \cos\theta_t}{Z_2 \cos\theta_i + Z_1 \cos\theta_t} \tag{22}$$

$$T_\perp = \frac{2 Z_2 \cos\theta_i}{Z_2 \cos\theta_i + Z_1 \cos\theta_t} \tag{23}$$

Before closing this subsection, two remarks regarding the previous equations should be stated. The first is concerned with the exponential factor in $\vec{H}_t$ which may be written explicitly as

$$e^{jk_2 \sin\theta_t x} e^{jk_2 \cos\theta_t z} \tag{24}$$

The first factor represents a plane wave propagating in the x-direction, which is from eqn (14) and equals

$$e^{jk_1 \sin\theta_i x} \tag{25}$$

The second factor, however, needs some caution since

$$\begin{aligned} k_2 \cos\theta_t &= k_2 \sqrt{1-\sin^2\theta_t} \\ &= k_2 \sqrt{1-\left(\frac{k_1}{k_2}\sin\theta_i\right)^2} \end{aligned} \tag{26}$$

which, for nonmagnetic media, may be written as

$$k_2 \cos\theta_t = k_2 \sqrt{1 - \left(\frac{n_1}{n_2}\sin\theta_i\right)^2} \tag{27}$$

where $n_j = \sqrt{\varepsilon_j}$ is the refractive index of the *j*th medium. This factor becomes purely imaginary whenever $\frac{n_1}{n_2}\sin\theta_i > 1$ and leads to an evanescent wave. This is commonly referred to as total internal reflection, and has a wide application in optical fibre links.[3] Another point of particular interest is to obtain a condition that enables the wave to proceed without any reflection, i.e., totally transmitted. This may be found by equating the numerator in eqn (22) to zero which leads, for nonmagnetic materials, to

$$\frac{\cos\theta_i}{\cos\theta_t} = \frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_t} \tag{28}$$

This is an impossible condition unless we have $n_1 = n_2$, i.e., the trivial case of a wave propagating in the same medium.

Parallel polarisation

The parallel polarisation (TM) case could be treated using the duality between the field quantities,[4] but we prefer to sketch the development in order to raise some hidden points and to show that the same general procedure may be carried out to derive the required coefficients. In this case, the incident magnetic field $\vec{H}_i$ is directed out of the page in the $\hat{a}_y$ direction. Using a similar notation, we may write

$$\vec{H}_i = \hat{a}_y H_{0i} e^{-j\vec{k}_{1i}\cdot\vec{r}} \tag{29}$$

$$\begin{aligned}\vec{E}_i &= \frac{\vec{H}_i \times \vec{k}_{1i}}{\omega\varepsilon_1} \\ &= Z_1 H_{0i}(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i) e^{-j\vec{k}_{1i}\cdot\vec{r}}\end{aligned} \tag{30}$$

$$\vec{H}_r = -\hat{a}_y H_{0r} e^{-j\vec{k}_{1r}\cdot\vec{r}} \tag{31}$$

$$\vec{E}_r = Z_1 H_{0r}(\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r) e^{-j\vec{k}_{1r}\cdot\vec{r}} \tag{32}$$

$$\vec{H}_t = \hat{a}_y H_{0t} e^{-j\vec{k}_t\cdot\vec{r}} \tag{33}$$

$$\vec{E}_t = Z_2 H_{0t}(\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t) e^{-j\vec{k}_t\cdot\vec{r}} \tag{34}$$

Imposing the boundary conditions and using eqns (14) and (15) to arrive at the following equations governing the fields' ratios

$$1 - \frac{H_{0r}}{H_{0i}} = \frac{H_{0t}}{H_{0i}} \tag{35}$$

$$Z_1 \cos\theta_i \left(1 + \frac{H_{0r}}{H_{0i}}\right) = Z_2 \cos\theta_t \frac{H_{0t}}{H_{0i}} \tag{36}$$

Solving these two equations results in

$$\frac{H_{0r}}{H_{0i}} = \frac{Z_2 \cos\theta_t - Z_1 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i} \tag{37}$$

$$\frac{H_{0t}}{H_{0i}} = \frac{2Z_1 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i} \tag{38}$$

These ratios may be defined as was done in eqns (18) and (19) but with respect to magnetic field intensities. This is what is usually done in transmission line theory to define a voltage and/or current reflection and transmission coefficients.[1] The preferred choice here is to stay with the previously defined ratios as Fresnel coefficients. The fields' amplitudes are related via the intrinsic wave impedances as

$$\begin{aligned} H_{0i} &= \frac{E_{0i}}{Z_1}, \\ H_{0r} &= \frac{E_{0r}}{Z_1}, \\ H_{0t} &= \frac{E_{0t}}{Z_2} \end{aligned} \tag{39}$$

Substituting in eqns (37) and (38) and using the definitions in eqns (18) and (19), but for parallel polarisation we obtain

$$\Gamma_{\parallel} = \frac{Z_2 \cos\theta_t - Z_1 \cos\theta_i}{Z_1 \cos\theta_i + Z_2 \cos\theta_t} \tag{40}$$

$$T_{\parallel} = \frac{2Z_2 \cos\theta_i}{Z_1 \cos\theta_i + Z_2 \cos\theta_t} \tag{41}$$

It is of interest, however, to note that at normal incidence ($\theta_i = 0$) the reflection and transmission coefficients are related, as may be deduced from eqn (36), via

$$1 + \Gamma_{\parallel} = T_{\parallel} \tag{42}$$

While for perpendicular polarisation

$$1 + \Gamma_{\perp} = T_{\perp}, \tag{43}$$

as suggested by eqn (17), and this is true for all θ_i.

The critical angle at which the wave may proceed without reflection, for this type of polarisation, is designated by θ_p and referred to as the polarisation or Brewster's angle. Setting the numerator of eqn (40) to zero will lead, upon using eqn (16) for nonmagnetic media, to

$$\sin\theta_p = \frac{1}{\sqrt{1 + (n_1/n_2)^2}} \tag{44}$$

or,

$$\theta_p = \tan^{-1}(n_1/n_2) \tag{45}$$

Illustrative examples

A MATLAB code given in Fig. 2 is used to produce three-dimensional snapshots of an appropriate component of a field quantity and their corresponding contours of

```
clear
close all
n1=input('n1=');
n2=input('n2=');
thinc=input('Incident angle=');
pol=input('Type of Polarization:  '' TE '' ,  ''TM'' :      ');
if pol=='TE'
   poll=1;
elseif pol=='TM'
   poll=2;
else
   error('Enter ''TE'' or ''TM'' ');
end
pol=poll;

i=0;
for zk1=-10:0.3:10
   i=i+1;

   z(i)=zk1;
   ii=0;
   for xk1=-10:.3:10
      ii=ii+1;

      switch pol
      case 1 % TE
         ey=TE(xk1, zk1,thinc ,n1,n2);
      case 2 %TM
         ey=TM(xk1, zk1,thinc,n1,n2);
      otherwise,
         disp('Incorrect Input')
      end
      x(ii)=xk1;
      f(i,ii)=imag(ey);
   end
end
[xn,yn]=meshgrid(z,x);
figure,meshc(yn,xn,f), xlabel('k_1 z'), ylabel('k_1 x')
figure,contour(yn,xn,f)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%   ****subroutine *******************
%   plane wave  obliquely incident upon planar dielectric interface
%  perpendicular polarization
%        inputs
%       xk1 = k1*x (x coordinate normalised by the wavenumber of
medium 1)
%       zk1 = k1*z (x coordinate normalised by the wavenumber of
medium 1)
%  output
% ey= complex phasor electric field at (xk1,zk1)
%
function ey=TE(xk1, zk1,thinc,n1,n2);
j=sqrt(-1.0);
%  Change from degrees to radians and take sine and cosine
        tinc= thinc*pi/180.0;
        sthinc=sin(tinc);
        cthinc=cos(tinc);
% ration of n2/n1 (indices of refraction)
rn21=n2/n1;
```

Fig. 2 *MATLAB script program.*

```
%  The x-components of the k-vectors are  multiplied by x.
%The z-components of the k-vectors are  multiplied by z.
        kixx=xk1*sthinc;
        kizz=zk1*cthinc;
%  test for total internal reflection (TIR)
   if (sthinc<=rn21) %propagating
         kz2=sqrt((1.-(sthinc/rn21).^2));
      end
        if (sthinc>rn21) %evancent
              kz2=-j*sqrt(((sthinc/rn21).^2-1.));
           end
% ktz*z and the normalisation
        ktzz=kz2*rn21*zk1;
% Fresnel reflection and transmission coefficients
        refl=(cthinc-rn21*kz2)/(cthinc+rn21*kz2);
        tran=2.*cthinc/(cthinc+rn21*kz2);
% in medium 1
if (zk1<= 0)
   zey=exp(-j*kizz)+refl*exp(j*kizz);
end
% in medium 2
if (zk1 > 0)
zey=tran*exp(-j*ktzz);
end
%  Now the common x-behavior
        ey=zey*exp(-j*kixx);

%%%%%%%%%%%   ****subroutine ******************
%   plane wave  obliquely incident upon planar dielectric
interface
%   parallel polarization
%       output
% ey= complex phasor magnetic field at (xk1,zk1)
 function ey=TM(xk1, zk1,thinc,n1,n2);
j=sqrt(-1.0);
        tinc= thinc*pi/180.0;
        stinc=sin(tinc);
        ctinc=cos(tinc);
        rn21=n2/n1;
        kixx=xk1*stinc;
        kizz=zk1*ctinc;
   if (stinc<=rn21)
         kz2=sqrt((1.-(stinc/rn21).^2));
      end
        if (stinc>rn21)
              kz2=-j*sqrt(((stinc/rn21).^2-1.));
           end
        ktzz=kz2*rn21*zk1;
% Reflection and transmission coefficients
        refl=(-rn21*ctinc+kz2)/(rn21*ctinc+kz2);
        tran=2.*rn21*ctinc/(rn21*ctinc+kz2);
if (zk1<= 0)
   zey=exp(-j*kizz)+refl*exp(j*kizz);
end
if (zk1 > 0)
            zey=tran*exp(-j*ktzz);
        end
        ey=zey*exp(-j*kixx);
```

Fig. 2 *Continued*

varying densities that are proportional to the magnitude of the field strength. We use dimensionless coordinates by choosing the wavenumber in the first medium k_1 and the velocity of light in free space as reference quantities; the time harmonic dependence $e^{j\omega t}$ is assumed throughout. In this section, some figures exemplifying the limiting cases discussed in the previous section will be presented. Figure 3 shows a case for perpendicularly polarised wave incident from an 'optically' dense medium of $n_1 = 4$ to another medium of $n_2 = 2$ with an incident wave angle $\theta_i = 50°$. We note that the wave is totally reflected to the first medium since $\theta_i > \sin^{-1}\left(\frac{n_2}{n_1}\right)$. The same is true for a TM case as Fig. 4 demonstrates. In the total internal reflection (Figs 3 and 4), the magnitude of the reflection coefficient equals unity as may be deduced from eqns (22) and (40) and checked from the MATLAB code. To show a case for a totally transmitted wave, which may occur only for a TM wave, let the incident medium be air (i.e., $n_1 = 1$) and the other medium glass of $n_2 = 1.52$; then if $\theta_i = \theta_p = 56.66°$ the wave will proceed without any reflection (i.e., $\Gamma_{\|} = 0$) as shown in Fig. 5. The transmitted wave, however, suffers from bending or refraction at the interface as may be noted from the contour plot of Fig. 6.

Conclusions

Illustrative figures produced using codes written in a widely used programming language for plane wave crossing an interface are given. The figures accurately picture the behaviour of an arbitrarily polarised wave incident obliquely at an interface

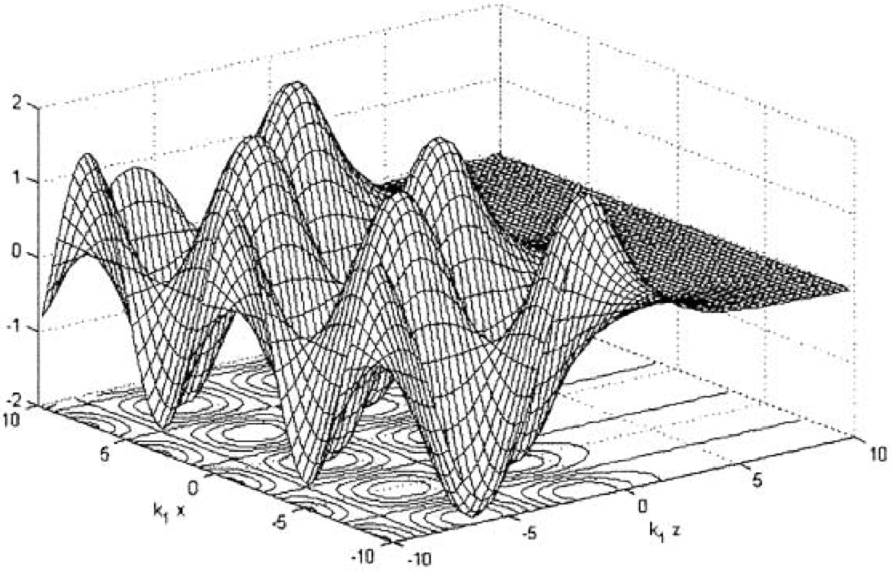

Fig. 3 *TE wave with* $n_1 = 4$, $n_2 = 2$, $\theta_i = 50°$.

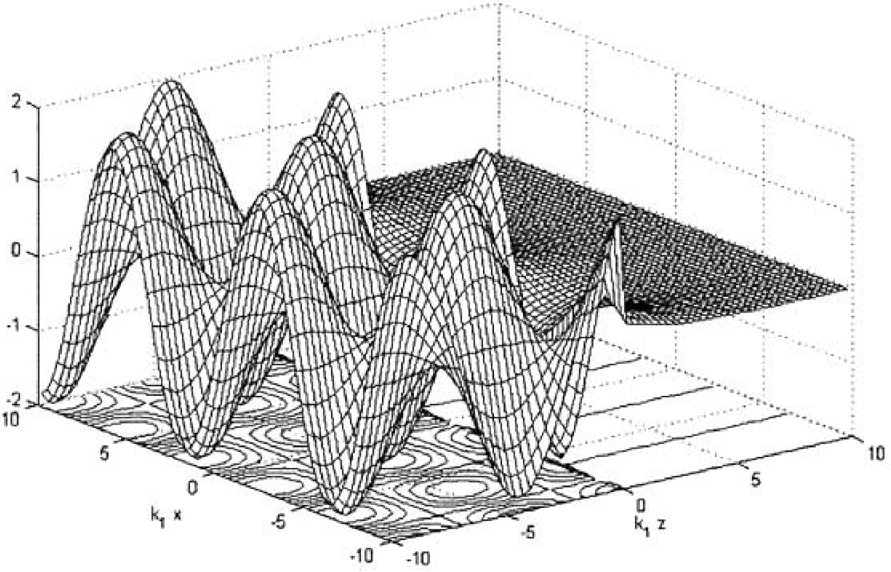

Fig. 4 *TM wave with* n$_1$ *= 4,* n$_2$ *= 2,* θ_i *= 50°.*

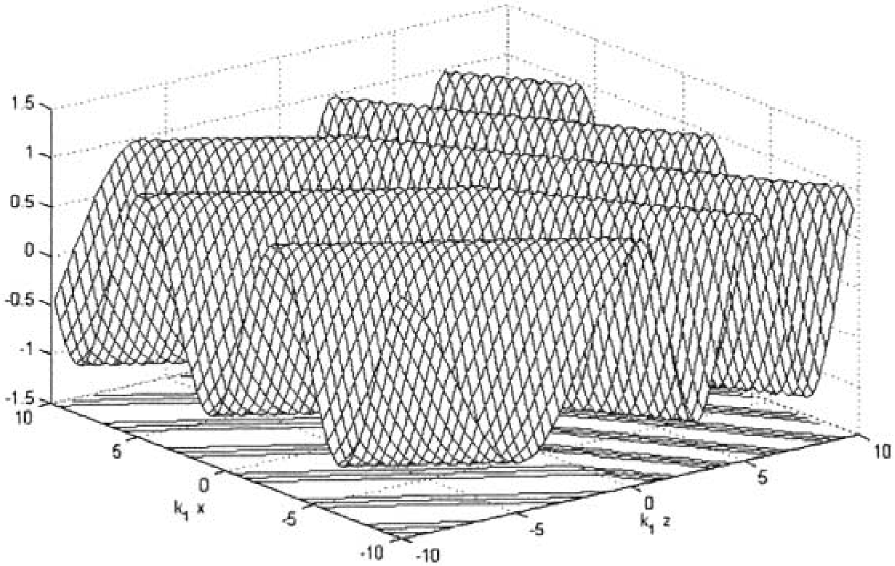

Fig. 5 *TM wave with* n$_1$ *= 1,* n$_2$ *= 1.52,* θ_i *= 56.66°.*

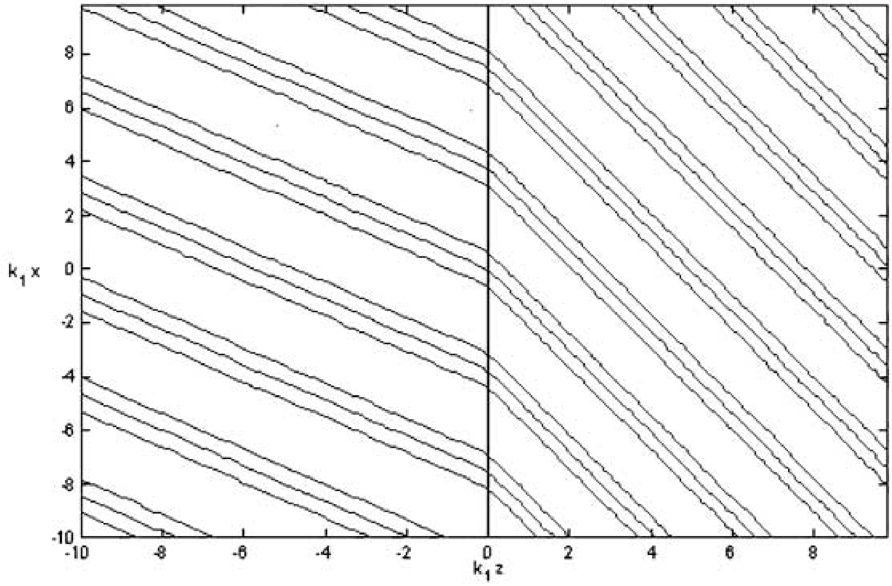

Fig. 6 *Contour plot of TM wave with* $n_1 = 1$, $n_2 = 1.52$, $\theta_i = 56.66°$.

separating two media without a restriction on their constitutive parameters. The approach used to derive the Fresenel coefficients is straightforward and applicable to both polarisations; it provides more insight into waves refracted at a planar interface and may be generalised to more complicated scenarios.

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