# Many faces of the maximum power transfer theorem

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**Abstract** The celebrated maximum power transfer theorem, which is usually stated for source impedance $Z_g = R_g + jX_g$ and load impedance $Z_L = R_L + jX_L$, is revisited, and generalised for other possible situations in which either $Z_g$ or $Z_L$ or both are parallel combinations of a resistance and a reactance, and for all possible cases of load variability. Many of the results are believed to be new.

**Keywords** circuit theory; matching; power transfer

## 1 Introduction

Figure 1 shows a sinusoidal voltage source of radian frequency $\omega$, r.m.s. value $V_g$ and fixed internal impedance $Z_g$ supplying power to a variable load $Z_L$. One would like to draw the maximum possible power from the source. Under what condition(s) is this possible? In most of the text books on circuit theory, $Z_g$ is taken as $R_g + jX_g$ and $Z_L$ as $R_L + jX_L$, and maximum power is shown to be absorbed by $R_L$ when $R_L = R_g$ and $X_L = -X_g$. This assumes that both $R_L$ and $X_L$ are variable. What happens when only $R_L$ or only $X_L$ or only $|Z_L|$ or only $\angle Z_L$ is variable? Such questions are often asked; the present paper provides answers for each of these cases.

In practice, $Z_g$ and $Z_L$ are not necessarily series combinations of resistance and reactance; either or both of them can be parallel combinations, instead. What are the conditions for maximum power transfer in these cases for the various possibilities of load variability? Comprehensive answers to these questions are also provided in this paper.

## 2 Series $Z_g$, series $Z_L$

The power absorbed by $R_L$, $P_L$, is given by

$$P_L = \left|V_g/(Z_g + Z_L)\right|^2 R_L = V_g^2 R_L \Big/ \left[(R_g + R_L)^2 + (X_g + X_L)^2\right]. \tag{1}$$

The various cases of load variability will be considered in the following subsections.

### 2.1 Only $X_L$ is variable

Equation (1) shows that if only $X_L$ is variable, then $P_L$ will be a maximum when

$$X_L = -X_g \tag{2}$$

and under this condition, $P_L$ becomes $P_{L1}$, where

$$P_{L1} = V_g^2 R_L \big/ (R_g + R_L)^2. \tag{3}$$

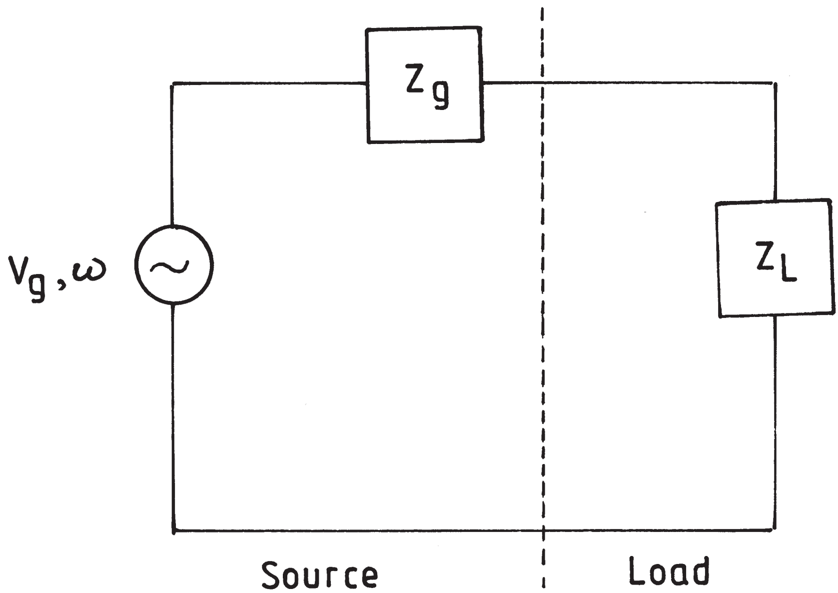


Fig. 1 *The circuit under consideration.*

### 2.2 Both $R_L$ and $X_L$ are variable

Once $X_L$ has been adjusted to maximise the power, one can now vary $R_L$ for further maximization. Note that eqn (3) can be rewritten as

$$P_{L1} = V_g^2 \Big/ \left[2R_g + R_L + \left(R_g^2/R_L\right)\right]. \tag{4}$$

The expression $Ax + B/x$, with $A$ and $B$ positive, is a minimum when $x = \sqrt{(B/A)}$. Hence $P_{L1}$ is maximum when

$$R_L = R_g \tag{5}$$

and under this condition, $P_{L1}$ becomes $P_{L2}$, where

$$P_{L2} = V_g^2 \big/ (4R_g). \tag{6}$$

### 2.3 Only $R_L$ is variable

To investigate this case, we rewrite eqn (1) as

$$P_L = V_g^2 \Big/ \left\{2R_g + R_L + \left[R_g^2 + \left(X_g + X_L\right)^2\right] \Big/ R_L\right\}. \tag{7}$$

By varying $R_L$, $P_L$ becomes a maximum when

$$R_L = \sqrt{R_g^2 + \left(X_g + X_L\right)^2}, \tag{8}$$

the maximum value being $P_{L3}$, where

$$P_{L3} = V_g^2 \Big/ \left\{2\left[R_g + \sqrt{R_g^2 + \left(X_g + X_L\right)^2}\right]\right\}. \tag{9}$$

Obviously, $P_{L3} < P_{L2}$; they become equal when $X_L = -X_g$, as expected.

### 2.4 Only $|Z_L|$ is variable

Let $Z_L = |Z_L| \exp(j\theta_L)$ and $Z_g = |Z_g| \exp(j\theta_g)$. Substituting $R_{L,g} = |Z_{L,g}| \cos\theta_{L,g}$ and $X_{L,g} = |Z_{L,g}| \sin\theta_{L,g}$ in eqn (1), and simplifying, we get

$$P_L = \frac{V_g^2 \cos\theta_L}{2|Z_g|\cos(\theta_L - \theta_g) + |Z_L| + \left(|Z_g|^2 / |Z_L|\right)}. \tag{10}$$

When $|Z_L|$ is varied, $P_L$ becomes a maximum when

$$|Z_L| = |Z_g|, \tag{11}$$

the corresponding value being $P_{L4}$, where

$$P_{L4} = \frac{V_g^2 \cos\theta_L}{2|Z_g|\left[1 + \cos(\theta_L - \theta_g)\right]}. \tag{12}$$

It can be easily verified that if $\theta_L$ is now varied, then $P_{L4}$ becomes a maximum when $\theta_L = -\theta_g$, and that under this condition, $P_{L4} = V_g^2/(4\,|Z_g| \cos\theta_g)$. Note that in the analysis in this subsection, we have not assumed any particular form of $Z_g$ or $Z_L$. The results are therefore general, independent of the configurations of $Z_g$ and $Z_L$. In the particular case of series $Z_g$, the maximum value of $P_{L4}$ becomes equal to $P_{L2}$ given by eqn (6), as expected.

### 2.5 Only $\theta_L = \angle Z_L$ is variable

Consider eqn (10) again, where except $\theta_L$, all other quantities are fixed. Differentiating $P_L$ with respect to $\theta_L$ and setting the result to zero gives, after simplification,

$$\sin\theta_L = -r\sin\theta_g, \tag{13}$$

where

$$r = 2\Big/\left(|Z_L|/|Z_g| + |Z_g|/|Z_L|\right) \le 1. \tag{14}$$

Under this condition, the value of $P_L$ becomes $P_{L5}$, where

$$P_{L5} = \frac{V_g^2}{2|Z_g|\left[\cos\theta_g + \sqrt{(1/r^2) - \sin^2\theta_g}\,\right]}. \tag{15}$$

Note that since $r \le 1$, $P_{L5}$ attains the maximum possible value of $V_g^2/(4\,|Z_g| \cos\theta_g)$ when $r = 1$ i.e. $|Z_L| = |Z_g|$. As in the previous subsection, the results of this subsection are also independent of the configurations of $Z_g$ and $Z_L$. For the particular case of series $Z_g$, the maximum value of $P_{L5}$ becomes equal to $P_{L2}$ given by eqn (6), as expected.

### 2.6 Discussion

It is to be noted that the case dealt with in subsection 2.2 is identical to that where $|Z_L|$ and $\angle Z_L$ are variable. In all other cases, the maximum possible power to the load cannot exceed $P_{L2}$. The cases discussed in subsections 2.1 and 2.3 may arise in many practical situations. The case discussed in subsection 2.4 in which only $|Z_L|$ is variable, keeping $\angle Z_L$ constant, is the constant Q situation and is also not unknown. However, the case of subsection 2.5 in which only $\angle Z_L$ is variable is not likely to be encountered in practice. We have included this case for the sake of completeness and for academic interest.

It is also emphasised that setting $Z_L = |Z_L| \exp(j\theta_L)$ and $Z_g = |Z_g| \exp(j\theta_g)$ does not make a distinction between series or parallel $Z_g$ or $Z_L$. Hence, as mentioned earlier, the results in subsections 2.4 and 2.5 will also be applicable to all cases discussed hereafter.

## 3 Series $Z_g$, parallel $Z_L$

Assuming $Z_g = R_g + jX_g$ and $Z_L = R_L \| jX_L$, it can be shown that the power absorbed by $R_L$ is given by

$$P'_L = \frac{V_g^2 X_L^2 R_L}{\left(R_L R_g - X_L X_g\right)^2 + \left[X_g R_L + X_L\left(R_g + R_L\right)\right]^2}. \tag{16}$$

We now consider the various situations that may arise. For brevity, we shall omit the steps for derivation and only state the condition(s) of maximum, give the value of the maximum power, and add comments, if any.

### 3.1 Only $X_L$ is variable

$$X_L = -\left|Z_g\right|^2 / X_g. \tag{17}$$

$$P'_{L1} = \frac{V_g^2 R_L}{\left|Z_g\right|^2 + 2R_g R_L + \left(R_g^2 R_L^2 / \left|Z_g\right|^2\right)}. \tag{18}$$

### 3.2 Both $R_L$ and $X_L$ are variable

$$X_L = -\left|Z_g\right|^2 / X_g \text{ and } R_L = \left|Z_g\right|^2 / R_g. \tag{19}$$

$$P'_{L2} = V_g^2 / \left(4R_g\right). \tag{20}$$

Note that eqn (20) is identical to eqn (6). *Thus the absolute maximum load power for series $Z_g$ is independent of whether $Z_L$ is a series or a parallel combination of resistance and reactance.*

### 3.3 Only $R_L$ is variable

$$R_L = \left|Z_g\right| \left|X_L\right| / \sqrt{R_g^2 + \left(X_g + X_L\right)^2}. \tag{21}$$

$$P'_{L3} = V_g^2 \left|X_L\right| / \left\{2\left[R_g \left|X_L\right| + \sqrt{R_g^2 + \left(X_g + X_L\right)^2} \left|Z_g\right|\right]\right\}. \tag{22}$$

Maximizing $P'_{L3}$ with respect to $X_L$ gives the same equations as (17) and (20), as it should.

## 4 Parallel $Z_g$, series $Z_L$

Assuming $Z_g = R_g \parallel jX_g$ and $Z_L = R_L + jX_L$, the power absorbed by $R_L$ becomes

$$P_L'' = \frac{V_g^2 R_L}{\left(R_L + |Zg|^2/R_g\right)^2 + \left(X_L + |Zg|^2/X_g\right)^2}. \tag{23}$$

### 4.1 Only $X_L$ is variable

$$X_L = -|Zg|^2/X_g. \tag{24}$$

Although this equation looks identical to eqn (17), they are not so, because in eqn (24), $|Zg| = Rg|Xg|/\sqrt{(Rg^2 + Xg^2)}$ while in eqn (17), $|Zg| = \sqrt{(Rg^2 + Xg^2)}$.

$$P_{L1}'' = \frac{V_g^2 R_L}{\left(R_L + |Zg|^2/R_g\right)^2}. \tag{25}$$

### 4.2 Both $R_L$ and $X_L$ are variable

$$X_L = -|Zg|^2/X_g \quad \text{and} \quad R_L = |Zg|^2/R_g. \tag{26}$$

Again, although eqns (26) and (19) look identical, they are not so because of the difference in the values of $|Z_g|$, as already mentioned.

$$P_{L2}'' = \left[V_g^2/(4R_g)\right]\left[1 + R_g^2/X_g^2\right]. \tag{27}$$

Note that $P_{L2}''$ is greater than $P_{L2}$ given by eqn (6).

### 4.3 Only $R_L$ is variable

$$R_L = \sqrt{\left(|Zg|^2/R_g\right)^2 + \left[X_L + \left(|Zg|^2/X_g\right)\right]^2}. \tag{28}$$

$$P_{L3}'' = \frac{V_g^2/2}{\left(|Zg|^2/R_g\right) + \sqrt{\left(|Zg|^2/R_g\right)^2 + \left[X_L + \left(|Zg|^2/X_g\right)\right]^2}}. \tag{29}$$

It can be verified that when $P_{L3}''$ is further maximised with respect to $X_L$, we get the same results as in eqns (24) and (27), as we should.

## 5 Parallel $Z_g$, parallel $Z_L$

This case is best analysed in terms of admittances, conductances and susceptances. Using standards notations, let $Y_{g,L} = G_{g,L} - jB_{g,L}$ where $G_{g,L} = 1/R_{g,L}$ and $B_{g,L} = 1/X_{g,L}$. Then the power absorbed in $G_L$ is

$$P_L''' = \frac{V_g^2 G_L\left(G_g^2 + B_g^2\right)}{\left(G_g + G_L\right)^2 + \left(B_g + B_L\right)^2}. \tag{30}$$

### 5.1 Only $B_L$ is variable

$$B_L = -B_g. \tag{31}$$

$$P_{L1}''' = \frac{V_g^2 G_L \left(G_g^2 + B_g^2\right)}{\left(G_g + G_L\right)^2}. \tag{32}$$

### 5.2 Both $G_L$ and $B_L$ are variable

$$G_L = G_g \text{ and } B_L = -B_g. \tag{33}$$

$$P_{L2}''' = \left(V_g^2 G_g/4\right)\left[1 + \left(B_g^2/G_g^2\right)\right]. \tag{34}$$

Equation (34) is the same as eqn (27). *Thus for parallel $Z_g$, the absolute maximum load power is the same for series as well as parallel $Z_L$, and this value exceeds that for series $Z_g$.*

### 5.3 Only $G_L$ is variable

$$G_L = \sqrt{G_g^2 + \left(B_g + B_L\right)^2}. \tag{35}$$

Note the similarity of this equation to eqn (8).

$$P_{L3}''' = V_g^2\left(G_g^2 + B_g^2\right)\Big/\left\{2\left[G_g + \sqrt{G_g^2 + \left(B_g + B_L\right)^2}\right]\right\}. \tag{36}$$

Maximizing $P_{L3}'''$ with respect to $B_L$, one gets eqns (31) and (34), as one should.

## 6 Plots of power ratio versus normalised variable

In all the cases considered so far, it is possible to find an expression for the ratio $p$ of actual load power to the maximum possible value in terms of a normalised value of the variable and, in some cases, one (or two) parameter(s). One can then draw what may be called universal plots. We shall illustrate this with the cases of section 2, viz. those for series $Z_g$ and series $Z_L$. With appropriate replacements, the same results can be made applicable to the case of section 5. The cases of sections 3 and 4 also have the feature of duality, and we have verified that such universal plots can be obtained for them also.

For the case when only $X_L$ is variable (subsection 2.1), the expression for $p$ is given by

$$p_{2.1} = P_L/P_{L1} = 1\Big/\left\{1 + \left[\left(X_g + X_L\right)/\left(R_g + R_L\right)\right]^2\right\}. \tag{37}$$

A plot of $p_{2.1}$ versus the normalised variable $(X_g + X_L)/ (R_g + R_L)$ is shown in Fig. 2.

If both $R_L$ and $X_L$ are variable (subsection 2.2) and eqn (2) has been satisfied, then the expression for $p$ takes the form

$$p_{2.2} = P_{L1}/P_{L2} = 4\Big/\left(\sqrt{R_L/R_g} + \sqrt{R_g/R_L}\right)^2. \tag{38}$$

This equation is plotted in Fig. 3 with $R_L/R_g$ as the variable.

With $R_L$ as the only variable we have $p_{2.3} = P_L/P_{L3}$ which can be simplified to the form

$$p_{2.3} = 2(g + 1)\big/\left[2g + y + (1/y)\right]. \tag{39}$$

where

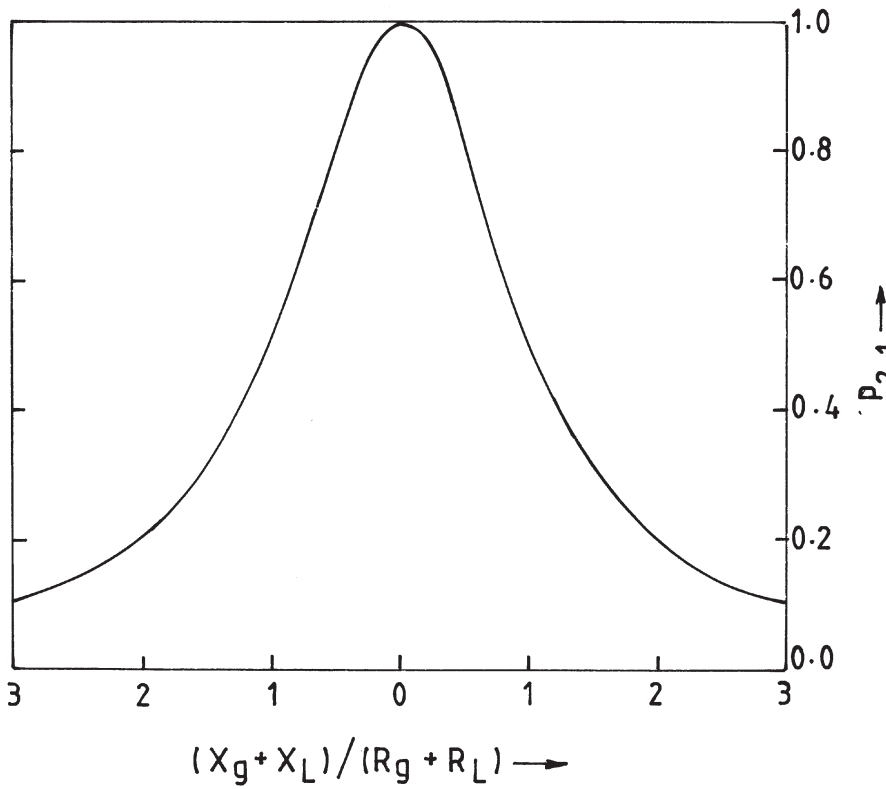


Fig. 2 *Universal plot for the case of subsection 2.1.*

$$g = R_g / R_{L0}, \tag{40a}$$

$$R_{L0} = \sqrt{R_g^2 + (X_g + X_L)^2}, \tag{40b}$$

and

$$y = R_L / R_{L0}. \tag{40c}$$

Plots of $p_{2.3}$ versus $y$ with $g$ as a parameter are shown in Fig. 4.

When $|Z_L|$ is variable the expression for $p_{2.4}$ can be easily put in the form of eqn (39) with

$$g = \cos(\theta_L - \theta_g) \tag{41a}$$

and

$$y = |Z_L| / |Z_g|. \tag{41b}$$

Thus the plots of Fig. 4 will also be applicable here.

Finally, when $\angle Z_L$ is the only variable we get, using eqns (10) and (15),

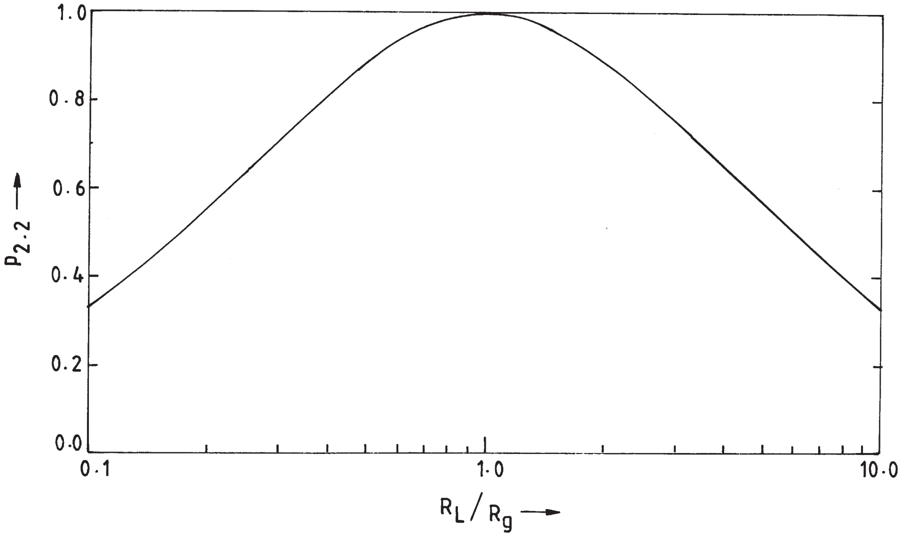


Fig. 3 *Universal plot for the case of subsection 2.2.*

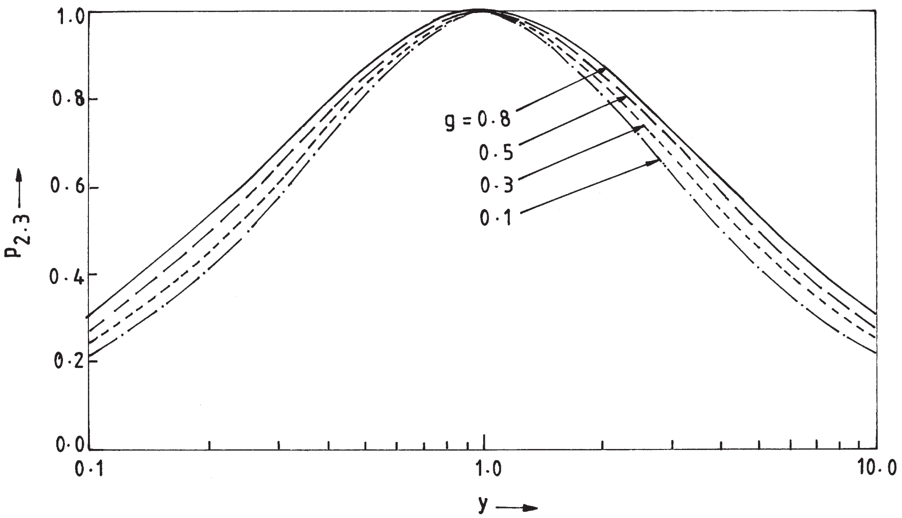


Fig. 4 *Universal plots for the case of subsection 2.3.*

$$p_{2.5} = \frac{\cos\theta_L\left[\cos\theta_g + \sqrt{(1/r^2) - \sin^2\theta_g}\right]}{\cos(\theta_L - \theta_g) + (1/r)}. \quad (42)$$

This expression can be put in the following form:

$$p_{2.5} = \frac{1 + \sqrt{1 - r^2\sin^2\theta_g}\Big/(r\cos\theta_g)}{1 + (1 - yr^2\sin^2\theta_g)\Big/\left[(r\cos\theta_g)\sqrt{1 - y^2r^2\sin^2\theta_g}\right]} \quad (43)$$

where

$$y = -\sin\theta_L\big/(r\sin\theta_g). \quad (44)$$

Obviously, $y = 1$ gives $p_{2.5} = 1$, as it should. Hence we need $\cos\theta_g$ (or $\sin\theta_g$) and $r$ as the two parameters to be able to draw plots of $p_{2.5}$ versus $y$. Since, as pointed out in subsection 2.6, this case is not of much practical importance, we shall not pursue it further.

## 7 Conclusions

In this paper, we have carried out a comprehensive investigation of the conditions for maximum power transfer to the load under various configurations of the source and load impedances and all possible situations of load variability. Three major new results have emerged. First, we have shown that with series $Z_g$, the maximum possible power to the load is $V_g^2/(4R_g)$ irrespective of whether the load is a series or a parallel combination of $R_L$ and $jX_L$. Secondly, with parallel $Z_g$, the maximum possible power to the load is greater than that for series $Z_g$ by the factor $[1 + (R_g^2/X_g^2)]$, irrespective of whether the load is a series or a parallel combination of $R_L$ and $jX_L$. Thirdly, when only $|Z_L|$ (or only $\angle Z_L$) is variable, the condition for, and the value of maximum power are given by eqns (11) and (12) [eqns (13) and (15)] respectively, irrespective of the configurations of $Z_g$ and $Z_L$.