Solving polynomial algebraic equations of the stand alone induction generator

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Abstract This paper describes the use of MathCAD's solving block which uses 'Given' and 'Find' built-in functions to solve *n*th order nonlinear algebraic equations. Introducing complex energy systems to electrical engineering students in their undergraduate studies is essential to complement many energy conversion courses. Various electric energy-capturing schemes use electric equivalent circuit models that incorporate nonlinear elements with complex mathematical formulas requiring numerical computation. Without incorporating programming tools, the taught material could be vague and a burden for both the student and the lecturer, hindering comprehension of the complexity of the system during the limited lecture hours. This paper introduces a 'ready to use' computational and mathematical tool that can be used to solve the non-linear equations quickly. The performance of an energy scheme with non-linear, interrelated variables such as the self-excited induction generator (SEIG) under variable excitation and loading conditions is used as an example. SEIG systems have been intensively proposed for energy capturing to supply power to remote areas from renewable energy resources such as wind and hydro prime mover systems.

Keywords computer education tools; induction generators; math solver tools, MathCAD

The computer classroom is a natural development of the traditional classroom. It allows 'learning-by-doing', a learning method that nearly doubles the effectiveness of 'seeing and hearing'. In recent years, simulation software packages have become an integral part of several electrical and computer engineering courses.[1–4] The advantages of using simulation tools to investigate an engineering system's physical behavior are tremendous. Several mathematical problems can be solved either symbolically or numerically using general-purpose mathematical software such as Matlab, Mathematica, Maple, and MathCAD.[5] The use of such software in the classroom and in the laboratory relieves the students from writing their own programs and reduces the time spent in understanding the physical meaning of the engineered systems. Computer simulation tools help students understand the physical concepts they study in the lectures and laboratory. However, it is widely acknowledged that computer simulation should not substitute actual laboratory practice, but rather they should complement each other. Electronic simulation software is a powerful tool and is proven to be accurate in resembling the physical behavior of many engineering devices. In some cases, the physical performance of the system depends on many variables that are interrelated. This makes it difficult to present the system by an equivalent circuit with defined passive or active elements. One of these systems, which is the main focus of the paper, is the induction generator used as a source of power in remote areas. An equivalent circuit can represent the induction generator for mathematical analysis. It becomes evident that the circuit elements are dependent on the generated frequency and voltage. Both the frequency and voltage are dependent on the internal magnetic characteristics and on the applied load at any particular prime mover speed. This makes the analysis of the generator performance difficult since the equivalent circuit analysis results in several nonlinear polynomial equations that are interrelated with each other.[6–8]

This paper introduces MathCAD software as an educational tool that can be used in the classroom as well as for research purposes. Equations are presented in the same way as they appear on a classroom board. All the necessary fraction bars, brackets and other characters are sized as they would be written on papers. Furthermore, MathCAD is sufficiently flexible and can be applied to a variety of teaching and learning situations. Some of the advantages of using MathCAD are:

- the students will write the equations in a familiar textbook format;
- the problems are solved interactively, which will boost the student's interest;
- the students will analyze the results and draw conclusions that enhance learning efficiency;
- the general-purpose math calculation programs are frequently used by industry;
- there is no need to learn dedicated, input-output type computer programs;
- the students will have a better understanding of the derivation of the generator characteristics from their equivalent circuit;
- student curiosity is stimulated to look for the computer model performance under different operating conditions.

In the following sections, two examples are presented. The focus is on the use of MathCAD's solving block in which 'Given' and 'Find' built-in functions are used to solve what could be a high-order nonlinear complex equation. A simple electric circuit is first used as an example to demonstrate the use of the solving block. The numerical results are then compared with PSpice circuit simulator to verify the method. The second example will apply the same algorithm in solving the non-linear polynomial equations of the self-excited induction generator and shows how it is practical to manipulate the equations with very minimal programming skills.

Computer-based circuit analysis using MathCAD solving block

Engineers routinely apply three types of computer analysis tool: a spreadsheet, a math solver and a circuit simulator. Each tool has its merits in cutting development costs and shortening analysis time. Spreadsheets are limited by how the scientific formulas can be manipulated and presented graphically. Spreadsheets are mainly used for business studies. Computer circuit analysis tools use an advanced graphical schematic editor that enables the user to draw a circuit diagram, which is then internally translated by the schematic capture to a netlist for processing. Using the devices model library that describes the inputted circuit creates a set of equations. Computer circuit analysis tools are very effective and result in high accuracy once the device model is correctly presented. For electromagnetic circuits such as the electric machine, only specialized and expensive software can accommodate the complexity of the system. This has led many educators and researchers to use mathematical software of affordable cost and easy usage to simulate the transient and the dynamic performance of such complex devices. By combining text, graphics and equations in a single document, math solvers software such as MathCAD make it easy to keep track of even the most complex calculation. Numerical results obtained from the math solvers and circuit analysis tools are very close as basically the solution algorithms are the same.

Many textbooks include examples that introduce the students to currently available simulation tools.[9] This encourages the students to practise the application of the computer simulation tools and helps them to understand the theory applied in the classroom. The following simple example is used to demonstrate how a math-solving tool can be used to solve unknown variables of a balancing bridge circuit inserted within the document in Fig. 1. Fig. 1 is a screen printout of a MathCAD document that presents the equations and constraints to solve for the unknown value of the midpoint voltage and current across resistor R3. Two sets of equations are formulated to present the circuit mathematically. Equating the algebraic sum of the currents, at any instant, at each node, makes the first set of equations equal to zero. Second sets of equations are written by equating the algebraic sum of voltage, at any instant, around the circuit loop to zero. The solution of the set of equations is conditioned by the constraints of the voltage drop across each element. This will give all possible values of the voltage and current in any branch in the circuit. In total, there are 12 equations that must be solved simultaneously to obtain the unknown values. Using the built-in solving block and the use of 'Given' and 'Find' built-in command, the solution block begins with the initial guesses for all unknown variables. A closer value of the initial guesses narrows the convergence of the solution of the unknown variables. The reserved word 'Given' tells MathCAD that what follows is a system of equations to be solved. The reserved word 'Find' ends the solve block. The constraint must be stated before the word 'Find' is typed. Mathcad iteratively solves the system equations listed between the 'Given' and 'Find' using the initial guesses listed before the 'Given' as a starting point. The result is then tabulated in matrix form, which represents the solution for the unknown under the stated constraints. The numerical results obtained in the example are compared to results of the same circuit configuration simulated using PSpice circuit analysis and are shown in Fig. 2. The comparison indicates very close agreement.

Using MathCAD to evaluate the stand alone self-excited generator variables

Since induction machines are not equipped with field windings to establish the required excitation magnetic field (the responsible media of converting and transfer energy), the excitation for a stand alone induction generator must be supplied externally. The external source for self-excitation is a capacitor bank across the machine terminals to provide the necessary leading current required to supply the machine and load reactive power requirements. The generated terminal voltage decreases with increasing load even when the speed is maintained at a constant value and sufficient capacitive energy is supplied. The voltage collapse is due to the non-

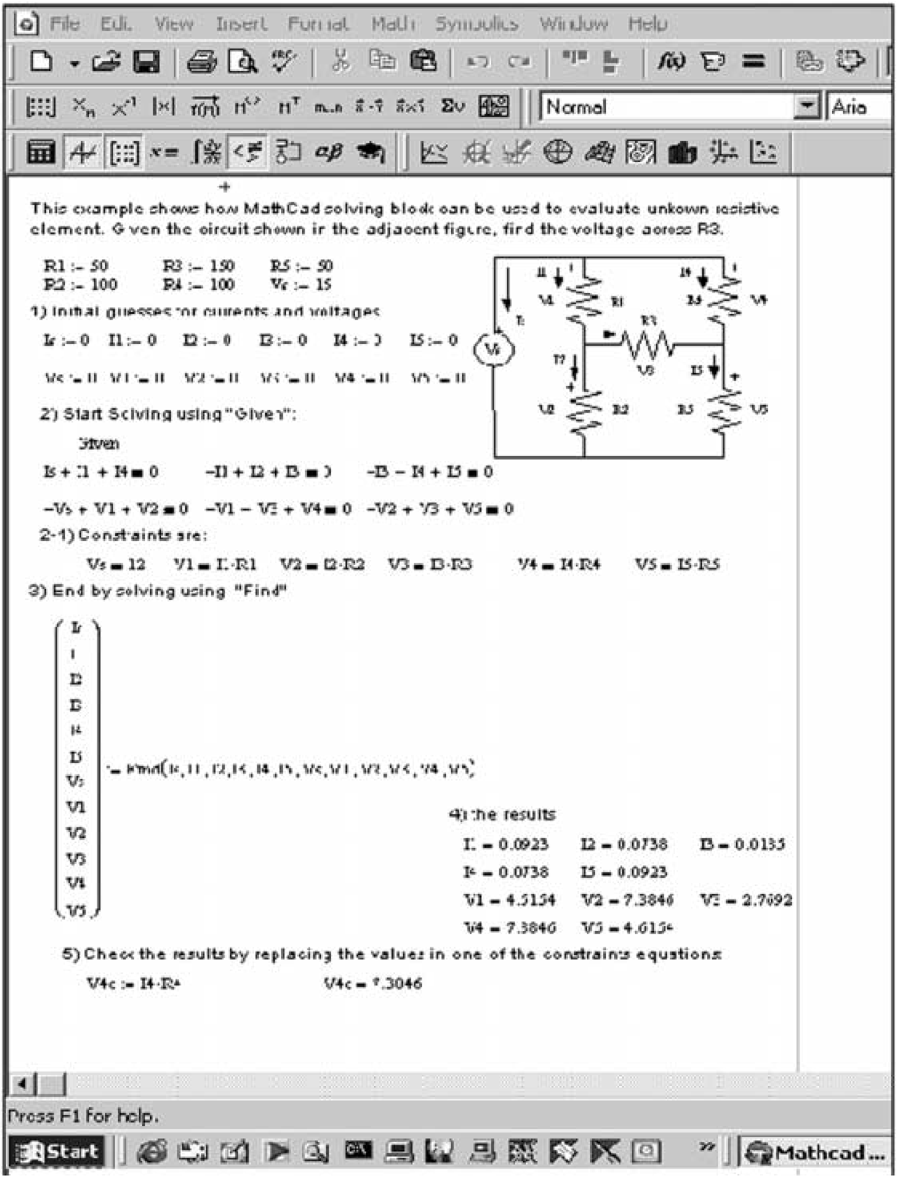


Fig. 1 *The use of MathCAD solving block 'Given' and 'Find'.*

linearity of the magnetizing element that supplies the magnetic flux within the generator and is affected by the internal impedance of the windings. The voltage is also sensitive to external variables such as the prime mover speed and the load size and nature. The same trend is followed by the generated frequency. This makes it more

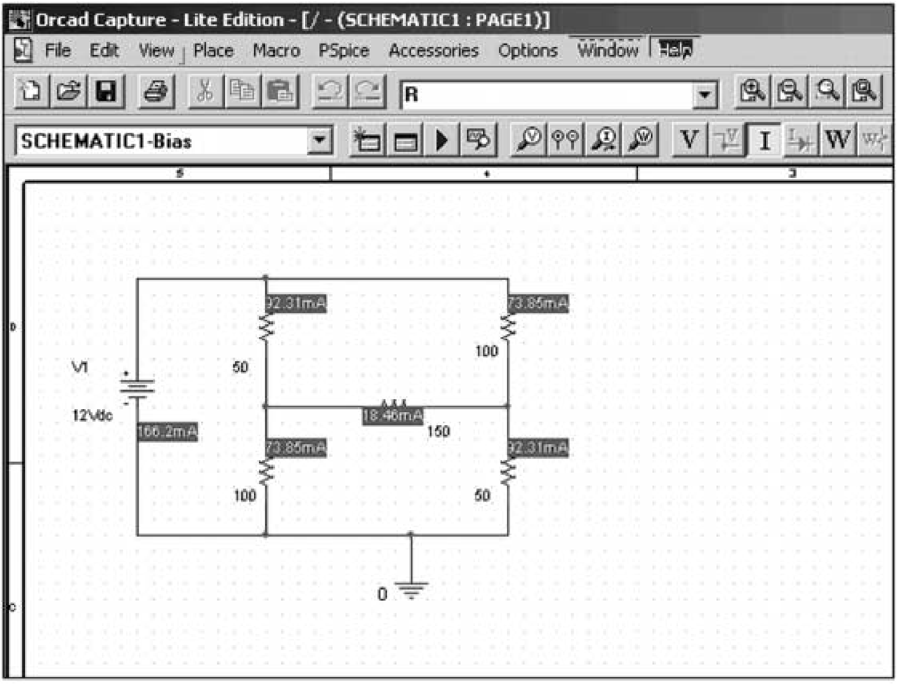


Fig. 2 *Simulation using Spice schematic software.*

difficult to predict the generator performance as compared to the case if the generator is connected to power grid in which the voltage and frequency are tied to the network.

Due to the complexity of the variables, symbolic mathematical presentation is needed to evaluate the machine magnetizing reactance X_m as a function of the rotor speed and frequency at the preset value of excitation reactance X_c. To initiate and sustain the voltage buildup in the generator windings, the generator magnetic and inductive reactive power with the load requirement must be matched by the external source. This means that the net volt-ampere within the generator equivalent circuit shown in the insert of Fig. 3 and the connected load must be equal to zero at any instant, otherwise the voltage will collapse. The total node admittance at the magnetizing element is zero:

$$Y_{LC1} + Y_m + Y_2 = 0 \tag{1}$$

where

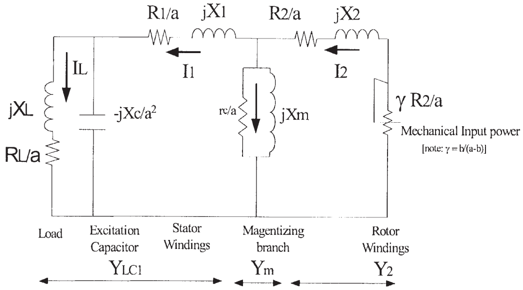
$$Y_{LC1} = 1\Big/\left[\left((R_L/a + jX_L)//-jX_c/a^2\right) + (R_1/a + jX_1)\right] \tag{2}$$

$$Y_m = (1/jX_m) + (a/r_c) \tag{3}$$

$$Y_2 = 1\big/(R_2/a + jX_2) \tag{4}$$

Given Data at the base frequency: *1.8kW, 60Hz, 380V Induction machine with the following passive elements measured at the base frequency:* $R_1 := 0.1.768$ *ohm* $R_2 := 2.91$ *ohm* $X_1 := 4.569$ *ohm* $X_L := 5.3$ *ohm* $R_L := 2.99$ *ohm* $X_2 := 4.569$ *ohm* $X_c := 51$ *ohm*

Initial guesses: $X_m := 82.19$ ohm $a := 1.01$ $b := 1.1$



Self-excited induction generator per phase equivalent circuit

<--------------------------------- Start

Given

Real part (YLC1 +Ym+Y2) = 0

$$\frac{\dfrac{\frac{RL\cdot XL\cdot Xc}{a^3}-\left[\frac{Xc\cdot RL}{a^3}\cdot\left(XL-\frac{Xc}{a^2}\right)\right]}{\left(\frac{RL}{a}\right)^2+\left(XL-\frac{Xc}{a^2}\right)^2}+\frac{R1}{a}}{\left[\dfrac{\frac{RL\cdot XL\cdot Xc}{a^3}-\left[\frac{Xc\cdot RL}{a^3}\cdot\left(XL-\frac{Xc}{a^2}\right)\right]}{\left(\frac{RL}{a}\right)^2+\left(XL-\frac{Xc}{a^2}\right)^2}+\frac{R1}{a}\right]^2+\left[\dfrac{\left(\frac{XL\cdot Xc}{a^2}\right)\cdot\left(XL-\frac{Xc}{a^2}\right)+\frac{Xc\cdot RL^2}{a^4}}{\left(\frac{RL}{a}\right)^2+\left(XL-\frac{Xc}{a^2}\right)^2}-X1\right]^2}+\frac{a}{rc}+\frac{\frac{R2}{(a-b)}}{\left[\frac{R2}{(a-b)}\right]^2+X2^2}=0$$

Imaginary part (YLC1+Ym+Y2) = 0

$$\frac{\dfrac{\left(\frac{XL\cdot Xc}{a^2}\right)\cdot\left(XL-\frac{Xc}{a^2}\right)+\frac{Xc\cdot RL^2}{a^4}}{\left(\frac{RL}{a}\right)^2+\left(XL-\frac{Xc}{a^2}\right)^2}-X1}{\left[\dfrac{\frac{RL\cdot XL\cdot Xc}{a^3}-\left[\frac{Xc\cdot RL}{a^3}\cdot\left(XL-\frac{Xc}{a^2}\right)\right]}{\left(\frac{RL}{a}\right)^2+\left(XL-\frac{Xc}{a^2}\right)^2}+\frac{R1}{a}\right]^2+\left[\dfrac{\left(\frac{XL\cdot Xc}{a^2}\right)\cdot\left(XL-\frac{Xc}{a^2}\right)+\frac{Xc\cdot RL^2}{a^4}}{\left(\frac{RL}{a}\right)^2+\left(XL-\frac{Xc}{a^2}\right)^2}-X1\right]^2}-\frac{1}{Xm}-\frac{X2}{\left[\frac{R2}{(a-b)}\right]^2+X2^2}=0$$

Constraints *Xc* $\geq$ *(1/Xm*$_{unsaturated}$*) and Xm = (Should be within limits)*
1< a < b, (a varies between 1 to 2)
b > 1 (b varies from the synchronous speed giving a ratio of 1 to 2),

$$\begin{pmatrix} a \\ b \\ Xm \end{pmatrix} := \mathrm{Find}(a, b, Xm)$$

Fig. 3 *Equations using Mathcad solving block to evaluate the unknown variables for the self-excited induction generator.*

The nodal admittance method of separating the real and imaginary components of the equivalent circuit, as described in Ref. [10], is used for numerical analysis. The conventional method of solving for the variables in the above equations is to equate the real and imaginary parts to zero. The resulting equations are then solved using a gradient method such as the Newton-Raphson method. This process is tedious and involves several algebraic manipulations that make it difficult if any of the variables are to be optimized. Furthermore, in the event of any changes in the connection of the load, the derived equations have to be re-arranged. Using MathCAD solving blocks 'Given' and 'Find' replaces the numerical step-by-step algebraic manipulation.[11] Only positive realistic values that meet the constraint criteria of energy balance will be accepted as a solution for the variables. The nodal admittance principle is used to separate the machine admittance into real and imaginary components, where equation (1) presents the real components while equation (2) presents the imaginary components. The two equations are simultaneously solved for the desired variable as demonstrated in Fig. 3.

The system equivalent circuit shown in Fig. 3 relates two variables. The first variable is 'a', the ratio relating the operating frequency and the base frequency. It depends on the rotating speed and has a dominant effect upon the operating slip and hence, the amount of power being transferred to the load. The solution of equation (1) will result in estimating the per unit frequency 'a'. The second variable is the magnetizing reactance X_m that is determined by the open circuit characteristics and is a function of the excitation capacitance X_c, load Z_L ($Z_L = R_L + jX_L$) and operating per unit speed ratio 'b'. The solution of equation (2) will result in estimating the magnetizing reactance X_m that gives the value of the air gap voltage induced across the generator terminal.

Both equations must satisfy the constraints of zero sums. Once the variables are evaluated, the air gap voltage E_m is estimated from the magnetizing curve, in which the values are collected experimentally. The procedures to solve the unknown variables 'a' and 'X_m' are as follows:

i Define all constants such as R_1, R_2, X_1, X_2, r_c, R_L and X_L in ohms. The terminal frequency is defined as a per unit ratio 'a' with the base frequency of 50 or 60 Hz. The rotor speed is also defined in per unit ratio 'b' with the base speed being 1500 or 1800 r.p.m. for a 4-pole machine.

ii Define the generator-magnetizing element X_m with the air gap induced voltage E_m. This must be obtained experimentally and can be represented in a piece-wise linearized form due to the non-linearity or as a polynomial curve-fitted equation:

$$E_m(X_m) = k_1 + k_2 X_m + k_3 X_m^2 + k_4 X_m^3 \ldots \quad (5)$$

where $E_m(X_m)$ is in per unit volt and the constants for the tested generator of a 1.8 kW, 380 V, and 60 Hz induction machine are:

$k_1 = -184.5267 \quad k_2 = 17.636$

$k_3 = -0.6236 \quad k_4 = 0.009$

iii Define the nodal admittances Y_{LC1}, Y_m and Y_2;
iv Give initial guess values for the variables $a = 1.01$, $b = 1.1$ and $X_m = 3$;
v Solve with 'Given' the nodal equations that are equated to zero and 'Find' the variables a, b and X_m that satisfy the conditions.

Fig. 3 illustrates the aforementioned procedures of entering data in a MathCAD sheet using the solving blocks of 'Given' and 'Find'. The induction generator is now open for investigation of the variables. For example, the required excitation capacitance to maintain constant voltage can be estimated for each load value. Also the effect of changing the rotor resistance can be investigated under the same constraints. This is simply carried out by changing the set of constraints and the matrix of unknown variables needed to be evaluated using the 'Find' solving word as shown in Fig. 4 in which R2 is now placed as a variable to maintain constant voltage.

To predict a set of data as the load varies, several programming loops are necessary to repeat the algorithm presented in Fig. 3 to cover the operating range required. Using cut and paste of the equations listed for the solutions of the variables and applying new sets of constraints and load values, new sets of values are generated. The procedures can be repeated in which the load is incremented with new constraints from the previous step. The range of data is then stored in a data matrix as shown in Fig. 5 that can be plotted for comparison in Fig. 6. The data matrix lists the generator terminal voltage and frequency as the load is varied with constant rotor resistance and excitation capacitor. A second set of data is also generated to maintain a constant terminal voltage by changing the rotor resistance using an external added resistor bank connected to the rotor windings through the slip rings of the generator.

Conclusion

The paper presents a teaching method that uses general-purpose mathematical software to solve a polynomial equation using a predefined simple procedure. The

New set of constraints

Constraints $Xc \geq (1/Xm_{unsaturated})$, and Xm = *fixed value*
$1 < a < b$, *(a varies between 1 to 2)*
$b > 1$ *(b varies from the synchronous speed giving a ratio of 1 to 2),*
$R2 \geq 0.577$

$$\begin{pmatrix} a \\ b \\ Xm \\ R2 \end{pmatrix} := \mathrm{Find}(a, b, Xm, R2)$$

Fig. 4 *Investigating the effect of changing the rotor resistance R2 by changing the constraints and the required matrix of unknowns.*

Data generated for constant excitation capacitance and constant rotor resistance:

AFr	AEr	AILr
1.1617	221.48926	1.75569
1.16593	244.05026	0.96418
1.16638	245.87757	0.86962
1.16759	250.36582	0.61107
1.16852	253.46766	0.40794
1.16957	256.66191	0.17486
1.16992	257.69044	0.09413
1.17012	258.25054	0.04894
1.17026	258.61774	0.01882
1.17031	258.77302	5.96657×10^{-3}

Data generated for constant excitation capacitance and variable rotor resistance to maintain constant voltage:

AF	AE	AIL
1.15635	220.05701	1.7514
1.05056	220.05701	0.95137
1.03917	220.05701	0.85991
1.00869	220.05701	0.6091
0.98527	220.05701	0.40992
0.95882	220.05701	0.17763
0.94975	220.05701	0.09602
0.9447	220.05701	0.05004
0.94133	220.05701	0.01927
0.9399	220.05701	6.11493×10^{-3}

Fig. 5 *Numerical results in which AE, AF and AIL are the voltage, frequency and load numerical results matrices respectively.*

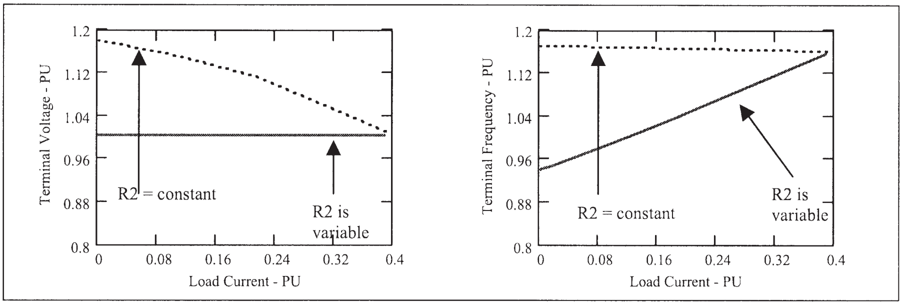


Fig. 6 *Graphical presentation of the results with the different constraints of Figs 3 and 4.*

procedure is quite simple and can be used effectively to solve many algebraic equations with interrelated multiple variables. The paper contributions are the following:

1 This paper describes the use of MathCAD's solving block that uses 'Given' and 'Find' built-in functions to solve *n*th order nonlinear complex induction machine equations.
2 The results of using MathCAD solving block 'Given' and 'Find' compare very well with the circuit simulation carried out using PSpice software.

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