Sychronisation of Chua's oscillator via the state observer technique

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Abstract Chua's oscillator is a simple autonomous third-order nonlinear circuit with a rich variety of dynamical behaviour. This paper presents a simple approach to synchronisation of Chua's oscillators using a state observer in control theory, which can be used as an interesting example for educational purposes in control systems, showing the students a simple way of coping with chaotic dynamics.

Keywords Chua's oscillator; control systems; synchronisation

Chaotic dynamics has become an unavoidable topic in some courses in Electrical Engineering Departments such as nonlinear circuit design and nonlinear control systems. With the availability of digital computers, the occurrence and nature of chaos can be demonstrated to the students. However, during the demonstration, the students usually ask such questions as: what is the use of chaos and can we exploit it? In fact, these questions are usually related to an active research topic in system and control engineering, which is the control and synchronisation of chaotic behaviour in nonlinear dynamical systems. In 1990, Ott, Grebogi and Yorke (OGY) suggested a method of controlling a chaotic system by stabilising one of the many unstable periodic orbits embedded in a chaotic attractor, using small time-dependent perturbations in the form of feedback to an accessible system parameter. Hence sensitive dependence on initial conditions was exploited to achieve control with a minimum control effort.[1] Control strategies such as linear state feedback and drive-response techniques have been synthesised which allow two or more chaotic systems to behave in a synchronised way.[2,3] However, these methodologies are too advanced to be introduced to those students who don't have particular interest in and specific knowledge of this topic.

In this paper particular attention will be devoted to the synchronisation of chaotic systems. Synchronisation is said to be achieved when two or more chaotic systems evolve in an identical manner. A number of applications have been proposed which require this problem to be solved, to exploit chaos for practical purposes. For example, in certain communication problems, the synchronisation of both a chaotic transmitter and chaotic receiver has been used to mask information signals. The signal that has to be transmitted is modulated by a chaotic carrier, so an unauthorised listener would not recognise the message but will just receive a noise-like signal. The transmitted signal can be decoded by a receiver only if the chaotic systems used for the codification and its parameters are both known. This method, called chaos shift keying,[4] was intended to yield a perfectly secure communication system via efficient synchronisation techniques.

In this paper, an observer-based approach for synchronising a typical chaotic system, Chua's oscillator, is presented. The Chua oscillator is one of the more well studied chaotic systems because of its simple structure. It has been shown to be a primitive system for studying chaos and bifurcation.[5,6] This example is quite acceptable to those students who only have fundamental knowledge of control systems and electronic circuits.

Chua's circuits

Chua's oscillator is a simple autonomous third-order nonlinear circuit with a rich variety of dynamical behaviour. It is composed of two capacitors, one inductor, two resistors and a piece-wise linear resistor as shown in Fig. 1. Its dimensionless state equations are:

$$\dot{x}_1 = \alpha(x_2 - h_0(x_1)) \tag{1}$$

$$\dot{x}_2 = x_1 - x_2 + x_3 \tag{2}$$

$$\dot{x}_3 = -\beta x_1 - \gamma x_3 \tag{3}$$

with

$$h_0(x_1) = m_1 x_1 + 0.5(m_0 - m_1)(|x_1 + 1| - |x_1 - 1|). \tag{4}$$

x_1, x_2 and x_3 being the state variables and α, β, γ, m_0, m_1 the system parameters. The well-known double scroll attractor is observed in Chua's oscillator if $\alpha = 9$, $\beta = 14.286$, $\gamma = 0.01$, $m_0 = -1/7$, $m_1 = 2/7$. The phase portrait in the x_1–x_2 plane is shown in Fig. 2(a), Another attractor is shown in Fig. 2(b), which can be obtained if $\alpha = 8.5$, $\beta = 14.286$, $\gamma = 0.01$, $m_0 = -1/7$, $m_1 = 2/7$.

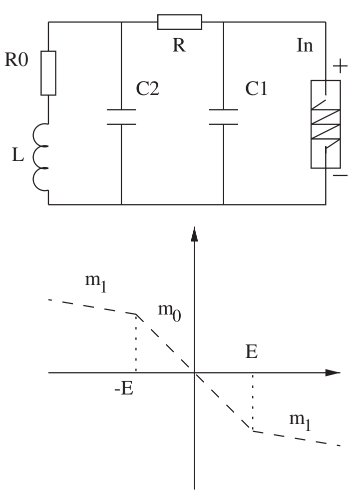

Fig. 1 *Chua's oscillator.*

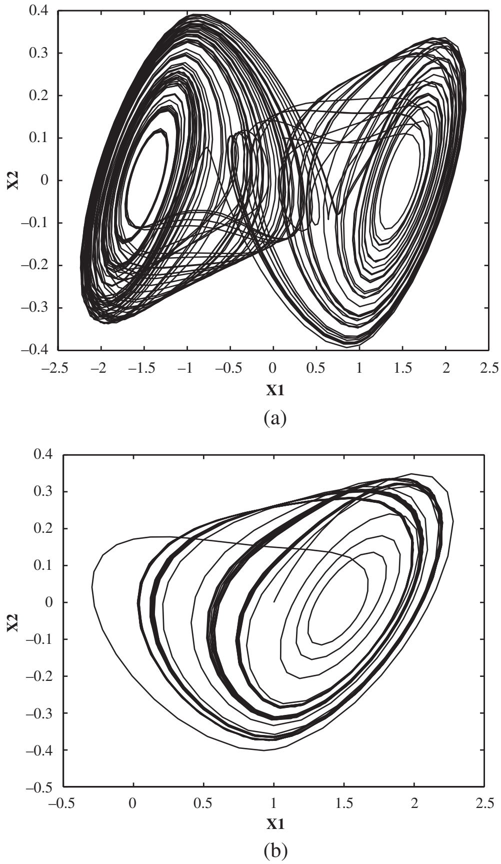

Fig. 2 *Strange attractors.*

Synchronisation of Chua's oscillator

Definition 1: Consider an autonomous n-dimensional dynamical system,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{5}$$

Divide the system, arbitrarily, into two subsystems $\mathbf{x} = (\mathbf{u}, \mathbf{v})^T$,

$$\dot{\mathbf{u}} = \mathbf{p}(\mathbf{u}, \mathbf{v}), \quad \dot{\mathbf{v}} = \mathbf{q}(\mathbf{u}, \mathbf{v}) \tag{6}$$

where $\mathbf{u} = (x_1, \ldots, x_m)$, $\mathbf{p} = [f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})]$, $\mathbf{v} = (x_{m+1}, \ldots, x_n)$, $\mathbf{q} = [f_{m+1}(\mathbf{x}), \ldots, f_n(\mathbf{x})]$. Now create a new subsystem $\mathbf{v}'$ identical to the $\mathbf{v}$ system, substitute the set of variables $\mathbf{u}$ for the corresponding $\mathbf{u}'$ in the function $\mathbf{q}$, and augment Eqns. (6) with this new system, giving

$$\dot{\mathbf{u}} = \mathbf{p}(\mathbf{u}, \mathbf{v}), \quad \dot{\mathbf{v}} = \mathbf{q}(\mathbf{u}, \mathbf{v}), \quad \dot{\mathbf{v}}' = \mathbf{q}(\mathbf{u}, \mathbf{v}'). \tag{7}$$

Examine the difference, $\Delta\mathbf{v} = \mathbf{v}' - \mathbf{v}$. The subsystem components $\mathbf{v}$ and $\mathbf{v}'$ will synchronize only if $\Delta\mathbf{v} \to 0$ as $t \to \infty$.

The essential property of a chaotic trajectory is that it is not asymptotically stable. Closely correlated initial conditions have trajectories which quickly become uncorrelated. Despite this obvious disadvantage, it has been established that the synchronisation of two chaotic systems is possible.

A popular synchronisation method that was first proposed by L. M. Pecora and T. L. Carroll is based on the hypotheses that the system that has to be synchronised can be decomposed into, at least, two subsystems. This technique is known as *the decomposition into subsystems method*.[2] The basic synchronisation procedure proposed by Pecora and Carroll can be described as follows.

Pecora and Carroll's approach

Suppose that the n-dimensional dynamical system (5) can be divided into two subsystems:

$$\dot{\mathbf{u}} = \mathbf{p}(\mathbf{u}, \mathbf{v}) \tag{8}$$

$$\dot{\mathbf{v}} = \mathbf{q}(\mathbf{u}, \mathbf{v}) \tag{9}$$

where $\mathbf{x} = (\mathbf{u}, \mathbf{v})^T$, $\mathbf{u} = (x_1, \ldots, x_m)$, $\mathbf{p} = [f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})]$, $\mathbf{v} = (x_{m+1}, \ldots, x_n)$, $\mathbf{q} = [f_{m+1}(\mathbf{x}), \ldots, f_n(\mathbf{x})]$.

Let us create a new subsystem $\hat{\mathbf{v}}$ identical to the $\mathbf{v}$ subsystem and augment (8, 9) with this new system, giving

$$\dot{\mathbf{u}} = \mathbf{p}(\mathbf{u}, \mathbf{v}) \tag{10}$$

$$\dot{\mathbf{v}} = \mathbf{q}(\mathbf{u}, \mathbf{v}) \tag{11}$$

$$\dot{\hat{\mathbf{v}}} = \mathbf{q}(\mathbf{u}, \hat{\mathbf{v}}) \tag{12}$$

the equations (10, 11) and the equation (12) are called a *driving system* and a *response subsystem*, respectively.

Lyapunov exponents of the response subsystem for a particular input $\mathbf{u}(\mathbf{t})$ are called *conditional Lyapunov exponents*. Let $\mathbf{v}(\mathbf{t})$ be a chaotic trajectory with initial condition $\mathbf{v}(\mathbf{0})$, and $\hat{\mathbf{v}}(\mathbf{t})$ be a trajectory started at a different initial point $\hat{\mathbf{v}}(\mathbf{0})$. It has been shown that the necessary condition for

$$\|\mathbf{v}(\mathbf{t}) - \hat{\mathbf{v}}(\mathbf{t})\| \to \mathbf{0}, \tag{13}$$

i.e., two subsystems to be synchronised, is that all of the conditional Lyapunov exponents are negative.[2]

We can describe this procedure using an example of Chua's circuit introduced in the above section. For the x_1-drive configuration, corresponding to $\mathbf{u} = x_1$, $\mathbf{v} = (x_2, x_3)$ and $\hat{\mathbf{v}} = (\hat{x}_2, \hat{x}_3)$ in (10)–(12), the state equations become:

(a) driving subsystem:

$$\dot{x}_1 = \alpha(x_2 - h_0(x_1)) \tag{14}$$

$$\dot{x}_2 = x_1 - x_2 + x_3 \tag{15}$$

$$\dot{x}_3 = -\beta x_1 - \gamma x_3 \tag{16}$$

(b) response subsystem:

$$\dot{\hat{x}}_2 = x_1 - \hat{x}_2 + \hat{x}_3 \tag{17}$$

$$\dot{\hat{x}}_3 = -\beta x_1 - \gamma \hat{x}_3 \tag{18}$$

It can be seen that for $\alpha = 8.5$, $\beta = 14.286$, $\gamma = 0.01$, $m_0 = -1/7$, $m_1 = 2/7$, driving and response subsystems can be synchronised, and the conditional Lyapunov exponents are ($\lambda_1^c = -1.0$, $\lambda_2^c = -0.01$).

State Observer approach

To achieve synchronisation of chaotic systems by Pecora and Carroll's approach, the conditional Lyapunov exponents of the subsystem are required to be negative. However, this is a necessary, but not sufficient condition for synchronisation. In order to obtain synchronisation, system (6) has to receive a proper synchronising signal from system (5). From a control theory point view, this signal can be considered as an observed quantity feeding a nonlinear observer for the state **x** of the system (5). Informally, an observer is a dynamic system designed to be driven by the output of the system (5) and having the property that the state of the observer converges to the state of the system (5). More precisely, the following definition is given.

Definition 2: Given the system (5) with output $\mathbf{z} = \mathbf{h}(\mathbf{x}) \in \mathbf{R}^{\mathbf{m}}$, the dynamic system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) + \mathbf{g}(\mathbf{z} - \mathbf{h}(\mathbf{y})), \quad (\mathbf{x}(\mathbf{0}), \mathbf{y}(\mathbf{0})) = (\mathbf{x}_0, \mathbf{y}_0) \tag{19}$$

is said to be a nonlinear observer of the system (5) if **y** converges to state **x** as $t \to \infty$, where $\mathbf{g}: \mathbf{R}^{\mathbf{m}} \to \mathbf{R}^{\mathbf{n}}$ is a suitably chosen nonlinear function.[7] Moreover, system (19) is said to be a global observer of system (1) if $\mathbf{y} \to \mathbf{x}$ as $t \to \infty$ for any initial condition $\mathbf{x}_0$, $\mathbf{y}_0$.[8] A block diagram of a nonlinear observer for state **x** of the system (5) is given in Fig. 3.

Let $\mathbf{e}(\mathbf{t})$ represent the observation error, $\mathbf{e}(\mathbf{t}) = \mathbf{x}(\mathbf{t}) \to \mathbf{y}(\mathbf{t})$, system (19) is a (global) observer of system (5) if the error system

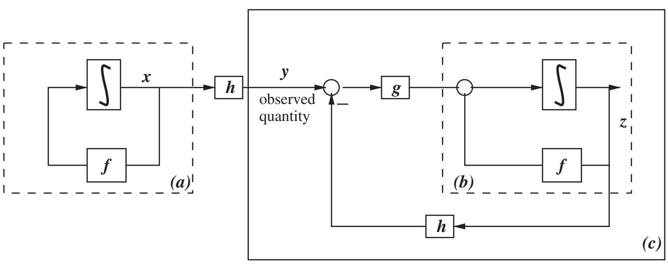

Fig. 3 *The structure of the nonlinear observer. (a) system (5); (b) observed system; (c) structure of the nonlinear observer (19).*

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{y})) - \mathbf{f}(\mathbf{y}) \\ &= \mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}+\mathbf{e})) - \mathbf{f}(\mathbf{x}+\mathbf{e}) \\ &= \mathbf{h_1}(\mathbf{e}, l)\end{aligned} \tag{20}$$

has a (globally) asymptotically stable equilibrium point for $\mathbf{e} = 0$.[7,8]

It is known that control theory offers no general method to choose a function $\mathbf{g}(\mathbf{z} - \mathbf{h}(\mathbf{y}))$ such that the nonlinear and nonautonomous system (20) has a (globally) asymptotically stable equilibrium point for $\mathbf{e} = \mathbf{0}$. However, for dynamic systems with specific system structure, it is sometimes possible to achieve synchronisation by choosing an appropriate transmitted signal $\mathbf{h}(\mathbf{x})$ and the function $\mathbf{g}$. This idea is illustrated using Chua's oscillator in the following.

For Chua's oscillator, we choose the scalar transmitted signal $\mathbf{h}(\mathbf{x}) \in \mathbf{R}$, i.e.,

$$\mathbf{h}(\mathbf{x}) = \alpha h_0(x_1) + \sum_{j=1}^{3} l_j x_j \in R.$$

and the function $\mathbf{g}$:

$$\mathbf{g}(\mathbf{h}(\mathbf{x}) - \mathbf{h}(\hat{\mathbf{x}})) = [-100]^T [\mathbf{h}(\mathbf{x}) - \mathbf{h}(\hat{\mathbf{x}})]$$

With the synchronisation setup shown in Fig. 3, the system (19) can be described as follows:

$$\dot{\hat{x}}_1 = \alpha(\hat{x}_2 - h_0(\hat{x}_1)) - \alpha h_0(x_1) + \alpha h_0(x_1) - \sum_{j=1}^{3} l_j (x_j - \hat{x}_j) \tag{21}$$

$$\dot{\hat{x}}_2 = \hat{x}_1 - \hat{x}_2 + \hat{x}_3 \tag{22}$$

$$\dot{\hat{x}}_3 = -\beta \hat{x}_2 - \gamma \hat{x}_3 \tag{23}$$

According to the drive system (eqns. 1–3) and the driven system (eqns. 21–23), the synchronisation error system is a linear time-invariant system described as:

$$\dot{e}_1 = \alpha e_2 + l_1 e_1 + l_2 e_2 + l_3 e_3 \tag{24}$$

$$\dot{e}_2 = e_1 - e_2 + e_3 \tag{25}$$

$$\dot{e}_3 = -\beta e_2 - \gamma e_3 \tag{26}$$

where $e_1 = x_1 - \hat{x}_1$, $e_2 = x_2 - \hat{x}_2$, $e_3 = x_3 - \hat{x}_3$.

Notably, if the system parameters $\alpha \neq 0$, $\gamma \neq 0$, the eigenvalues of the system (eqns. 24–26) can be arbitrarily assigned by appropriately selecting the gains l_i, i = 1, 2, 3 due to the observability of the system. For instance, for $\alpha = 8.5$, $\beta = 14.286$, $\gamma = 0.01$, $m_0 = -1/7$, $m_1 = 2/7$, the eigenvalues of eqns. (24–26) are placed in –1.0956 + 2.5769 i, –1.0956 – 2.5769 i, and –1.8189 for $l_1 = -3.0$, $l_2 = -3.0$, $l_3 = -2.0$. As the real parts of all eigenvalues of the error system are negative, therefore, all state errors converge to zero exponentially, implying that system dynamics eqns. (1–3) and eqns. (21–23) are synchronised exponentially. For those mentioned parameters,

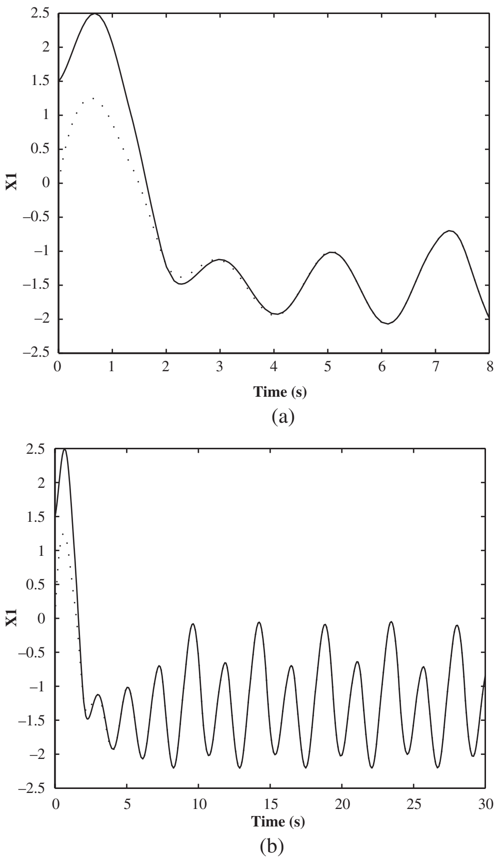

Fig. 4 *Synchronisation performance of Chua's oscillator (initial conditions and the parameters are chosen as:* ($x_1(0)$, $x_2(0)$, $x_3(0)$, $\hat{x}_1(0)$, $\hat{x}_2(0)$, $\hat{x}_3(0)$) = *(1.5, 0.1, 0, 0, 0, 0),* $\alpha = 8.5$, $\beta = 14.286$, $\lambda = 0.01$, $m_o = -1/7$, $m_1 = 2/7$.*)*

i.e., $\alpha = 8.5$, $\beta = 14.286$, $\gamma = 0.01$, $m_0 = -1/7$, $m_1 = 2/7$, Figs. 4 and 5 show the synchronisation performances under different initial condition, with different time scales. Both figures indicate that the state variable $\hat{x}_1$ tracks x_1 after the transient period. It has also been observed that the synchronisation performances are guaranteed with other different initial conditions.

Conclusion

A simple approach to synchronising Chua's oscillator with chaotic dynamics is presented. The approach employs a state observer in control theory and can be used as an interesting example for educational purposes in control systems and electronic circuits.

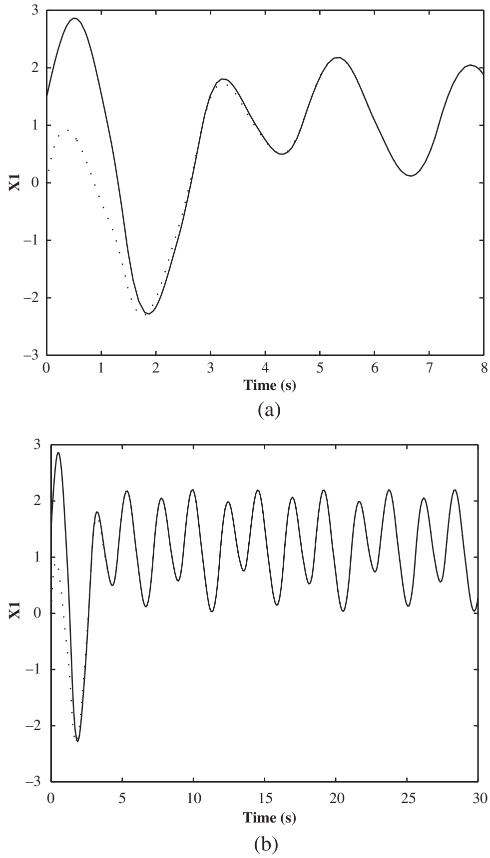

Fig. 5 *Synchronisation performance of Chua's oscillator (initial conditions and the parameters are chosen as:* $(x_1(0), x_2(0), x_3(0), \hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0)) = (1.5, 0.5, 0.1, 0, 0, 0)$, $\alpha = 8.5$, $\beta = 14.286$, $\lambda = 0.01$, $m_o = -1/7$, $m_1 = 2/7$.)

Acknowledgment

The authors would like to thank the referee for his valuable comments and appreciate the financial support from the National Natural Science Foundation of China, (Project No. 60074040) the Teaching and Research Award Program for Outstanding Young Teachers (TRAPOYT) in Higher Education Institutions, People's Republic of China.

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