The complex transformer as a network-model element

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Abstract The transformer representation of invariance of power is extended via the concepts of complex, vector, matrix and pseudo-transformers.

Keywords network model; complex transformer; transformer

Equivalent circuits, or in more modern terms network models, have long been used to represent the performance of electromagnetic energy-conversion devices such as transformers, induction machines, etc. The reason for this is that the network model presents the relationships between the variables in more easily 'readable' and memorisable form than the corresponding mathematical equations.

In the network representation the transformer as a network model element represents invariance of power. The conventional equivalent circuit for a power transformer of Fig. 1(a) for instance, consists of three impedances $\mathbf{Z}_a$, $\mathbf{Z}_b$, $\mathbf{Z}_c$, in principle all complex, including resistive loss components, and an N : 1 network model ideal transformer. Since the latter is conceived as an idealised physical device – i.e. what the physical transformer would be were it not for its 'imperfections' of core reluctance, leakage, and losses – its ratio N is always real.

Morris[1] derived an alternative general model for a physical transformer consisting of two complex impedances **Z** and a network model ideal transformer with a complex ratio **N**. His model is shown in slightly modified form in Fig. 1(b). (The reason for the double slashes will be discussed below.) The present paper develops further the concept of a complex transformer, extending the range of mathematical relations that can be represented in network model form.

General complex transformer

First consider a general complex transformer as a network model element. From now on, unless otherwise stated, the term 'transformer' always refers to the network-model element made of ink-on-paper, as opposed to its physical counterpart made of copper, steel, insulation, etc.

Defining the VA (voltamps) **s** at a point in a 2-wire network with complex potential **u** and flow **i**, and the polarities of Fig. 2(a)

$$\mathbf{s} = \mathbf{u}\mathbf{i}^* \tag{1}$$

a general network-model transformer can be defined

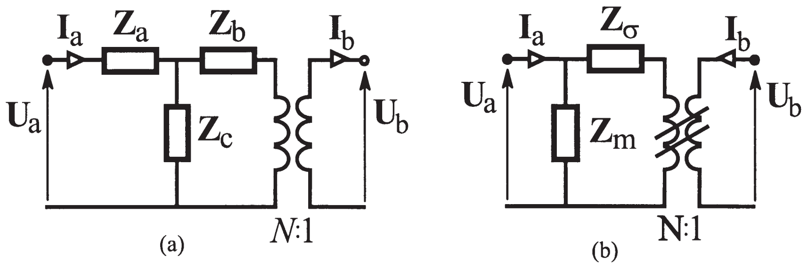


Fig. 1 *Power transformer models with (a) real, (b) complex network-model transformers.*

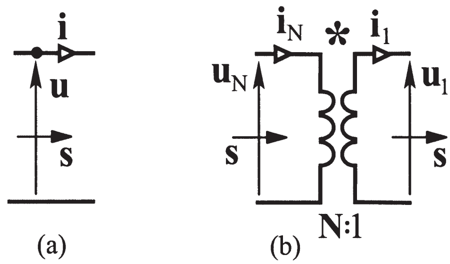


Fig. 2 *(a) VA. (b) Network-model transformer.*

a transformer is a network-model element representing invariance of VA

and represented as in Fig. 2(b).

Consider such a complex transformer with potentials and flows $\mathbf{u}_N$, $\mathbf{i}_N$ on the **N**-side and $\mathbf{u}_1$, $\mathbf{i}_1$ on the 1-side, Fig. 2(b). The invariance-of-VA condition gives

$$\mathbf{s} = \mathbf{u}_N\mathbf{i}_N{}^* = \mathbf{u}_1\mathbf{i}_1{}^* \qquad (2)$$

whence

$$\frac{\mathbf{u}_N}{\mathbf{u}_1} = \frac{\mathbf{i}_1{}^*}{\mathbf{i}_N{}^*} = \mathbf{N} \qquad (3)$$

where the complex constant **N** is the transformation ratio of the transformer. Any value of **N** satisfies the invariance-of-VA condition.

Writing eqs (3) in the form

$$\mathbf{u}_N = \mathbf{N}\mathbf{u}_1; \qquad \mathbf{i}_1 = \mathbf{N}^*\mathbf{i}_N \qquad (4)$$

a general network model transformer can be regarded as representing two independent relationships, namely a potential/potential relationship and a flow/flow relationship. For a complex transformer the corresponding transformation ratios are thus not the same, the flow ratio **N*** being the complex conjugate of the potential ratio **N**. So we will adopt the convention of always labelling a transformer with its potential ratio (**N**).

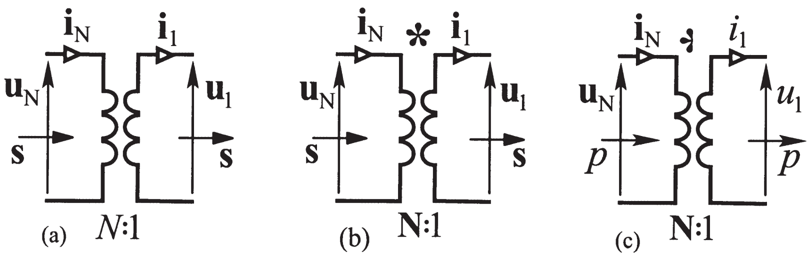


Fig. 3 *(a) Real, (b) complex, (c) complex:real transformers.*

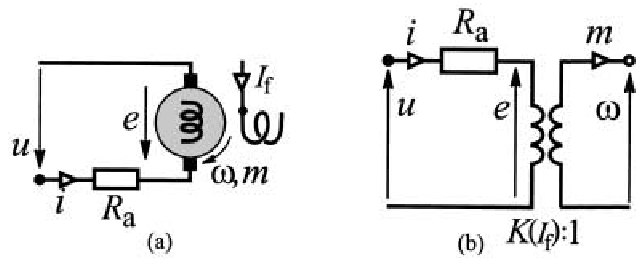


Fig. 4 *D.c. machine. (a) Schematic representation. (b) Circuit model.*

If required a complex transformer can be distinguished from its real counterpart by a '*', real and complex transformers being shown in Figs 3(a,b).

Complex:real transformer

Another possible transformer form is a complex:real transformer, Fig. 3(c), representing relationships between a complex potential **u** and flow **i** on the **N**-side, and a real potential u and flow i on the 1-side. The potential/potential and flow/flow relationships represented by a complex:real transformer are defined

$$\mathbf{u}_N = \mathbf{N}\mathbf{u}_1; \qquad i_1 = \mathcal{Re}\{\mathbf{N}^*\mathbf{i}_N\} \tag{5}$$

In this case it is the real part of the VA **s** – i.e. the power p – that is invariant across the transformer

$$p = \mathcal{Re}\{\mathbf{u}_N\mathbf{i}_N^*\} = u_1 i_1 \tag{6}$$

Real and complex:real transformers can be used for example to represent the relations between the electric and mechanical variables of d.c. and a.c. machines.

For instance, for the semi-ideal d.c. machine represented schematically in Fig. 4(a), with no loss other than the armature-resistive loss, the excitation/speed (*e/ω*) and torque/current (m/i) relations can be written

$$e = K(I_f)\omega; \qquad m = K(I_f)i \tag{7}$$

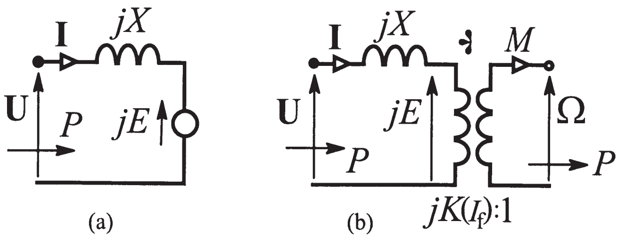


Fig. 5 *Synchronous machine. (a) Electrical, (b) electromechanical phasor models.*

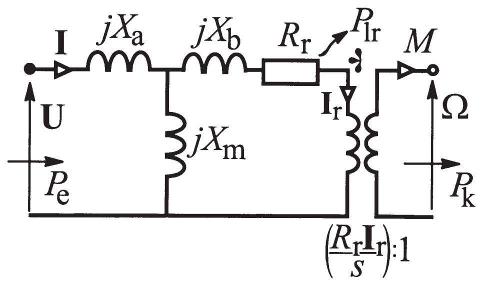


Fig. 6 *Induction machine. Electromechanical model.*

where $K(I_f)$ is a function of the field current I_f. Including the armature resistance R_a the machine can then be represented by the circuit model of Fig. 4(b) with a $K(I_f)$:1 real transformer.

Similarly, for the *synchronous machine* with electrical phasor model as Fig. 5(a) and excitation/speed (**E**/Ω) and torque/current (M/**I**) relationships

$$\mathbf{E} = jE = jK(I_f)\Omega; \qquad M = \mathcal{R}e\{jK(I_f)\mathbf{I}^*\} \tag{8}$$

The invariance-of-power condition between the electrical and mechanical sides is here (cf eqn. 6)

$$P = \mathcal{R}e\{\mathbf{EI}^*\} = \Omega M \tag{9}$$

So the electromechanical conversion is represented by a $jK(I_f)$:1 complex:real transformer as in Fig. 5(b).

The standard induction machine phasor model can be redrawn as in Fig. 6. The power in element R_r represents the rotor resistive loss P_{lr}, and the power through the ($R_r\mathbf{I}_r/s$):1 complex:real transformer represents the electromechanical conversion.

The complex:real transformer with a current- and slip-dependent ratio evidently has no direct physical correlate. However that is no problem, since a network-model transformer is a conceptual element representing a mathematical relationship between variables.

And so on. Basically any form of energy-conversion under invariance of power, and where the power is the product of potential and flow variables, can be represented in transformer form.

A transformer in a network-model effectively says ‘conversion under invariance of power’, and helps emphasise in visual form the essential unity of all electromechanical conversion devices.

Sequence components

Defining phase and sequence voltages ($\mathbf{U}$), ($\mathbf{U}_\$$) of a 3-phase system

$$(\mathbf{U}) = \begin{pmatrix} \mathbf{U}_a \\ \mathbf{U}_b \\ \mathbf{U}_c \end{pmatrix}; \qquad (\mathbf{U}_\$) = \begin{pmatrix} \mathbf{U}_0 \\ \mathbf{U}_I \\ \mathbf{U}_{II} \end{pmatrix} \tag{10}$$

and similarly phase and sequence currents ($\mathbf{I}$), ($\mathbf{I}_\$$), the relationships between the two can be defined

$$(\mathbf{U}) = [\mathbf{h}^*](\mathbf{U}_\$); \qquad (\mathbf{I}_\$) = \frac{1}{3}[\mathbf{h}](\mathbf{I}) \tag{11}$$

where the matrix [$\mathbf{h}$]

$$[\mathbf{h}] = \begin{bmatrix} 1 & 1 & \mathbf{1} \\ 1 & \mathbf{h} & \mathbf{h}^* \\ 1 & \mathbf{h}^* & \mathbf{h} \end{bmatrix}; \qquad \mathbf{h} = e^{j2\pi/3} \tag{12}$$

The phase voltages ($\mathbf{U}$) in terms of the sequence voltages ($\mathbf{U}_\$$) are then

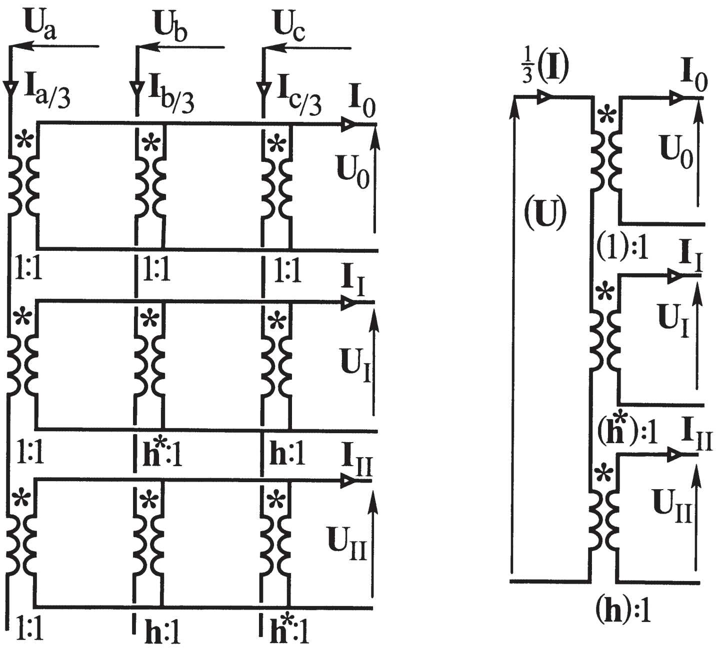
$$\begin{aligned} \mathbf{U}_a &= \mathbf{U}_0 + \mathbf{U}_I + \mathbf{U}_{II} \\ \mathbf{U}_b &= \mathbf{U}_0 + \mathbf{h}^*\mathbf{U}_I + \mathbf{h}\mathbf{U}_{II} \\ \mathbf{U}_c &= \mathbf{U}_0 + \mathbf{h}\mathbf{U}_I + \mathbf{h}^*\mathbf{U}_{II} \end{aligned} \tag{13}$$

And similarly the sequence currents ($\mathbf{I}_\$$) in terms of the phase currents ($\mathbf{I}$) are

$$\begin{aligned} \mathbf{I}_0 &= \frac{1}{3}(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) \\ \mathbf{I}_I &= \frac{1}{3}(\mathbf{I}_a + \mathbf{h}\mathbf{I}_b + \mathbf{h}^*\mathbf{I}_c) \\ \mathbf{I}_{II} &= \frac{1}{3}(\mathbf{I}_a + \mathbf{h}^*\mathbf{I}_b + \mathbf{h}\mathbf{I}_c) \end{aligned} \tag{14}$$

These relationships are represented in network form via the complex transformers of Fig. 7(a).

The factor ‘1/3’ applied to the phase currents is due to the definition of sequence currents ($\mathbf{I}_\$$) adopted (eqn. 11), in turn due to the requirement that unit phase power correspond to unit sequence power.



(a) (b)

Fig. 7 *(a) 3-phase sequence components. (b) Ditto, using vector transformers.*

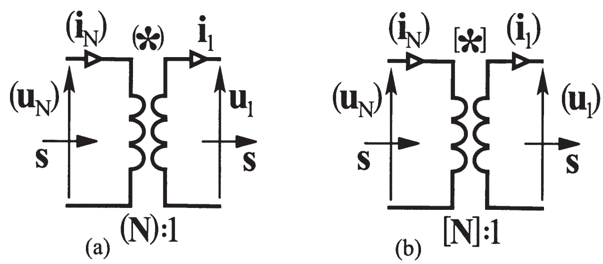


Fig. 8 *(a) Vector, (b) matrix transformers.*

Vector transformer

The sequence relations of Fig. 7(a) can be represented more concisely by defining an (**N**):1 vector transformer, drawn as in Fig. 8(a), as an element representing the relationships (cf eqn. 4)

$$(\mathbf{u}_N) = (\mathbf{N})\mathbf{u}_1; \qquad \mathbf{i}_1 = \{\mathbf{N}^*\}(\mathbf{i}_N) \tag{15}$$

where the vector ratio (**N**)

$$(\mathbf{N}) = \begin{pmatrix} \mathbf{N}_a \\ \mathbf{N}_b \\ \mathbf{N}_c \\ \cdots \end{pmatrix} = \{\mathbf{N}\}^t \tag{16}$$

So a vector transformer relates vector variables ($\mathbf{u}_N$), ($\mathbf{i}_N$) on the **N**-side to single variables $\mathbf{u}_1$, $\mathbf{i}_1$ on the 1-side, both in general complex. The flow/flow ratio {**N***} is here the conjugate transpose of the potential/potential ratio (**N**). It can be checked that the VA **s**

$$\mathbf{s} = \{\mathbf{i}^*\}(\mathbf{u}) = \mathbf{u}\mathbf{i}^* \tag{17}$$

is invariant across the coupler.

Using such vector transformers the sequence circuit representation of Fig. 7(a) reduces to that of Fig. 7(b), where the vector ratios

$$(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \qquad (\mathbf{h}^*) = \begin{pmatrix} 1 \\ \mathbf{h}^* \\ \mathbf{h} \end{pmatrix}; \qquad (\mathbf{h}) = \begin{pmatrix} 1 \\ \mathbf{h} \\ \mathbf{h}^* \end{pmatrix} \tag{18}$$

Matrix transformer

A still further economy of representation in the sequence-circuit case can be achieved defining an [**N**]:1 matrix transformer, Fig. 8(b), as representing the relationships (cf eqn. 15)

$$(\mathbf{u}_N) = [\mathbf{N}](\mathbf{u}_1); \qquad (\mathbf{i}_1) = [\mathbf{N}^*]^t(\mathbf{i}_N) \tag{19}$$

The flow/flow ratio $[\mathbf{N}^*]^t$ is again the conjugate transpose of the potential/potential ratio [**N**] (cf eqn. 15). It can again be checked that the VA **s**

$$\mathbf{s} = \{\mathbf{i}_N^*\}(\mathbf{u}_N) = \{\mathbf{i}_1^*\}(\mathbf{u}_1) \tag{20}$$

is invariant across the coupler.

Using a matrix transformer the sequence circuits of Fig. 7(b) reduce to those of Fig. 9.

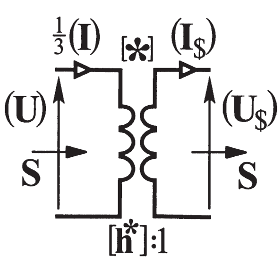


Fig. 9 *Sequence quantities, matrix-transformer representation.*

The complex matrix transformer is the most general form of network-model transformer, covering all preceding forms.

Pseudo-transformer

The last element to be considered is a pseudo-transformer, Fig. 10(a), defined as representing the relationships (cf eqn. 3)

$$\frac{\mathbf{u}_N}{\mathbf{u}_1} = \frac{\mathbf{i}_1}{\mathbf{i}_N} = \mathbf{N} \tag{21}$$

A pseudo-transformer thus has the same complex ratio **N** for both potential/potential and flow/flow relationships

$$\mathbf{u}_N = \mathbf{N}\mathbf{u}_1; \qquad \mathbf{i}_1 = \mathbf{N}\mathbf{i}_N \tag{22}$$

So a pseudo-transformer effectively represents the same relationships as a real transformer, except replacing the real ratio N by the complex ratio **N**. Unlike the complex transformer (cf eqn. 4) the potential/potential and flow/flow ratios are here the same. Writing the ratio **N**

$$\mathbf{N} = \mathrm{N}e^{jv} \tag{23}$$

the N-side VA in terms of the 1-side VA (eqn. 22)

$$\mathbf{s}_N = \mathbf{u}_N\mathbf{i}_N^* = e^{j2v}\mathbf{s}_1 \tag{24}$$

So across a pseudo-transformer the VA **s** and hence power p are not invariant – i.e. the pseudo-transformer is not a true transformer. Hence the name ‘pseudo-’ and the representation with a double slash.

For a real ratio **N** = N (v = 0), the pseudo-transformer reduces to a real transformer.

Abc-redundancy

The use of a complex **N**:1 pseudo-transformer eliminates the well known redundancy of the ‘abc’ transformer model of Fig. 1(a), namely that there are in principle an infinite number of possible models of this form representing the mathematical relationship between the terminal variables of the device. This is reflected in the fact that its parameter values cannot be uniquely determined via measurements at its terminals alone.

Consider a general 2-coil system, Fig. 10(b), with relation between the voltage **U** and current **I** variables given by the mathematical model

$$\begin{pmatrix} \mathbf{U}_a \\ \mathbf{U}_b \end{pmatrix} = \begin{bmatrix} \mathbf{Z}_{aa} & \mathbf{Z}_{ab} \\ \ldots & \mathbf{Z}_{bb} \end{bmatrix} \begin{pmatrix} \mathbf{I}_a \\ \mathbf{I}_b \end{pmatrix} \tag{25}$$

Consider also the corresponding ‘σm’ network model of Fig. 1(b). Comparing first the side A oc-impedances of the mathematical and circuit models, on the math-

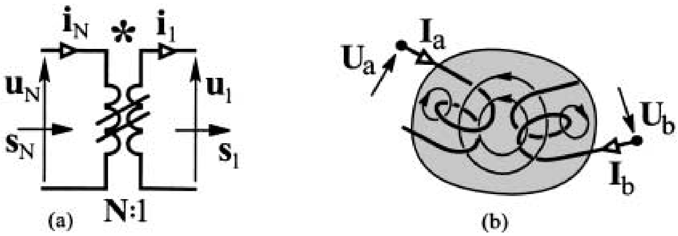


Fig. 10 *(a) Pseudo-transformer. (b) General 2-coil system.*

ematical model this is the self-impedance $\mathbf{Z}_{aa}$ and on the σm-model it is the shunt impedance $\mathbf{Z}_m$. Whence the model impedance $\mathbf{Z}_m$

$$\mathbf{Z}_m = \mathbf{Z}_{aa} \tag{26}$$

Now consider the potential ratio $\mathbf{U}_a/\mathbf{U}_b$ with side A supplied and side B open-circuited. On the mathematical model this is the ratio $\mathbf{Z}_{aa}/\mathbf{Z}_{ab}$ and on the circuit model it is the pseudo-transformer ratio **N**. Whence the model ratio **N**

$$\mathbf{N} = \frac{\mathbf{Z}_{aa}}{\mathbf{Z}_{ab}} \tag{27}$$

And considering the side B oc impedance, on the mathematical model this is $\mathbf{Z}_{bb}$ and on the σm-model it is $(\mathbf{Z}_m + \mathbf{Z}_\sigma)/\mathbf{N}^2$. Whence the model impedance $\mathbf{Z}_\sigma$

$$\mathbf{Z}_\sigma = \mathbf{N}^2\mathbf{Z}_{bb} - \mathbf{Z}_m \tag{28}$$

So based on the three impedance-matrix impedances $\mathbf{Z}_{aa}$, etc. (eqn. 25), the three σm-model parameters $\mathbf{Z}_m$, $\mathbf{Z}_\sigma$, **N** are uniquely determined, and the redundancy of the 4-parameter abc-model (Fig. 1(a)) is avoided.

Conclusion

Extending the 'ideal transformer' concept to elements with complex and vector ratios, a wider variety of mathematical relationships can be represented in network model form.

Reference

1 D. G. O. Morris, 'Transformer equivalent circuit', *Proc. IEE*, **97**(II) (1950), 17, 735.