The centre of twist for a prismatic bar under free torsion

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Abstract This paper presents a proof that the centre of twist for a prismatic bar under torsion coincides with the centroid of the cross-section. The centre of twist has long been regarded as arbitrary, since the consequence of a differently chosen centre of twist amounts only to a rigid-body rotation. However, what has not been properly addressed is that an arbitrarily chosen centre of twist results in an inconsistency in the formulation of conventional torsion theory.

Keywords torsion; centre of twist; centroid; prismatic bar; St Venant problem; warping

The classic St Venant problem of torsion of prismatic bars is formulated based on the observation of separable warping displacement and displacements in the plane of the cross-section corresponding to a rigid-body rotation. In many textbooks on elasticity [e.g. 1–5], the formulation of the problem assumes *a priori* the coincidence of the origin of the coordinate system in the plane of the cross-section with the centre of twist, without a further discussion of the location of the centre of twist. In several classic textbooks on elasticity [e.g. 6–9], it has been suggested that the location of the centre of twist is arbitrary and determined only by how the ends are fixed. The difference between different selections of the centre of twist amounts to a rigid-body rotation in space. It would seem that no point in a cross-section can be regarded as better than another as the centre of twist. Many authors have suggested aligning the centre of twist with the shear centre of the cross-section [e.g. 7, 10–13].

However, an arbitrarily chosen centre of twist causes a small difficulty in the consistency of the formulation of the problem, although it does not affect the calculated stress distribution. In the conventional formulation of the problem, the displacements are introduced as a rotation in the plane of the cross-section about the longitudinal axis and a warping displacement. The separation of the deformation into these two components is possible only if the longitudinal axis is located at a unique position, as will be discussed later. This point is the *true centre of twist*, as opposed to an arbitrarily chosen one. This true centre of twist can be perceived as follows.

The formulation of the torsion problem requires only a small (unit) length of the bar, and the angle of twist is measured over this length. Within this length, all the longitudinal fibres tilt a bit (except one) after torsion deformation, but remain approximately straight under the small deformation assumption. However, when many of these unit lengths are put together, these fibres will eventually form helices. The only exception is the fibre passing through the centre of twist, which remains perfectly straight. In this sense, the centre of twist is unique for a given cross-section. The displacements in the plane of a cross-section result from a rigid-body rotation about that point. A natural question to ask is where this centre of twist is. Chou and Pagano [14] stated that the centre of twist varies with the shape of the cross-section of the bar, but can it be found before the torsion problem is solved? Prescott [15] suggested that it was the centroid, but the justification was neither clear nor rigorous. In this paper, a rigorous proof will be presented that the true centre of twist indeed coincides with the centroid of the cross-section in general.

Conventional formulation of the torsion problem with a warping function

When the origin of the coordinate system is located at the true centre of twist of a prismatic bar under free torsion, the displacement field is given as:

$$u = -\alpha yz$$
$$v = \alpha xz$$
$$w = \alpha\psi(x, y) \tag{1}$$

where α is the angle of twist per unit length of the bar and ψ is the warping function. The shear strains can be obtained from the above displacements as:

$$\gamma_{xz} = \alpha\left(\frac{\partial\psi}{\partial x} - y\right) \quad \text{and} \quad \gamma_{yz} = \alpha\left(\frac{\partial\psi}{\partial y} + x\right) \tag{2}$$

Shear stresses are:

$$\tau_{xz} = G\alpha\left(\frac{\partial\psi}{\partial x} - y\right) \quad \text{and} \quad \tau_{yz} = G\alpha\left(\frac{\partial\psi}{\partial y} + x\right) \tag{3}$$

The governing equation (equilibrium) for the problem is:

$$\nabla^2\psi = 0 \quad \text{in} \quad A \tag{4}$$

where A is the cross-section area. The governing equation is subjected to the following boundary condition.

$$\frac{\partial\psi}{\partial x}dy - \frac{\partial\psi}{\partial y}dx = \frac{1}{2}d\left(x^2 + y^2\right) \quad \text{along } \partial A \tag{5}$$

where ∂A is the boundary of A.

The formulation above can be found in almost all textbooks on elasticity. It is consistent only if the origin of the x–y plane is at the true centre of twist. Although the centre of twist does not appear in the formulation, knowledge of the position of the centre of twist is required before it is applied.

Torsion problem formulated with an arbitrarily located origin

Assume the true centre of twist is C, which is different from the origin of coordinates, O, as shown in Fig. 1. By definition, as the bar deforms, point C remains stationary in the x–y plane. The in-plane displacements corresponding to the rotation can therefore be given as:

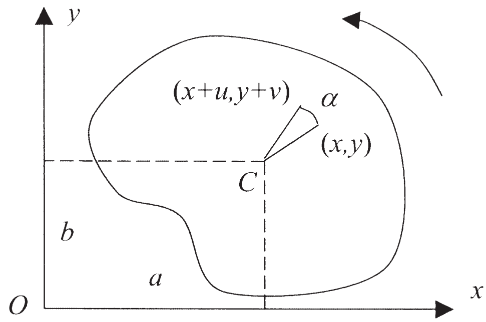

Fig. 1 *Cross-section of a prismatic bar under torsion.*

$$u = -\alpha(y-b)z \quad \text{and} \quad v = \alpha(x-a)z \tag{6}$$

while the expression of the warping displacement remains the same as in equation (1), where *a* and *b* are the coordinates of *C*.

Shear strains and stresses can be obtained correspondingly as:

$$\gamma_{xz} = \alpha\left(\frac{\partial\psi}{\partial x} - y + b\right) \quad \text{and} \quad \gamma_{yz} = \alpha\left(\frac{\partial\psi}{\partial y} + x - a\right)$$

$$\tau_{xz} = G\alpha\left(\frac{\partial\psi}{\partial x} - y + b\right) \quad \text{and} \quad \tau_{yz} = G\alpha\left(\frac{\partial\psi}{\partial y} + x - a\right) \tag{7}$$

The governing equation remains the same as in equation (4), but the boundary condition changes to:

$$\frac{\partial\psi}{\partial x}dy - \frac{\partial\psi}{\partial y}dx + adx + bdy = \frac{1}{2}d\left(x^2 + y^2\right) \quad \text{along } \partial A \tag{8}$$

The problem above explicitly requires the position of the centre of twist, *C*, i.e. coordinates *a* and *b*, in the boundary condition (equation 8). Can the location of the centre of twist be made known before a torsion problem is solved? This will be addressed in the next section.

Stress resultants and the true centre of twist

For a pure torsion problem, the shear forces, as the resultants of shear stresses over the cross-section, should vanish. This is:

$$\begin{aligned} Q_x &= \iint_A \tau_{xz} dA = G\alpha \iint_A \left(\frac{\partial\psi}{\partial x} - y + b\right) dA \\ &= G\alpha\left(\psi_x - S_x + bA\right) = G\alpha\left(\psi_x - S_x^c\right) = 0 \\ Q_y &= \iint_A \tau_{yz} dA = G\alpha \iint_A \left(\frac{\partial\psi}{\partial y} - x - a\right) dA \\ &= G\alpha\left(-\psi_y + S_y - aA\right) = -G\alpha\left(\psi_y - S_y^c\right) = 0 \end{aligned} \tag{9}$$

where:

$$\psi_x = \iint_A \frac{\partial \psi}{\partial x} dA \quad \text{and} \quad \psi_y = -\iint_A \frac{\partial \psi}{\partial y} dA \tag{10}$$

$$S_x = \iint_A y dA \quad \text{and} \quad S_y = \iint_A x dA \tag{11}$$

$$S_x^c = S_x - bA \quad \text{and} \quad S_y^c = S_y - aA \tag{12}$$

S_x and S_y are the moments of the cross-section area of the bar about the x and y axes and S_x^c and S_y^c are the moments of the same about axes parallel to the x and y axes but passing through point C. Vanishing of shear forces results in the following relationships:

$$\psi_x - S_x^c = 0 \quad \text{and} \quad \psi_y - S_y^c = 0 \tag{13}$$

Consider now the torque as the resultant of shear stresses over the cross-section:

$$T = \iint_A (\tau_{yz} x - \tau_{xz} y) dA = G\alpha \iint_A \left(x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} + x^2 + y^2 - ax - by \right) dA$$

$$= G\alpha \left[\iint_A \left(x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} \right) dA + J_p - aS_y - bS_x \right] \tag{14}$$

where $J_p = \iint_A (x^2 + y^2) dA$ is the polar moment of the cross-section area about the origin, O, of the coordinate system.

Using the Green's theorem and the boundary condition (equation 8), following the same procedure as in [16], the integral in equation (14) can be transformed to:

$$\iint_A \left(x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} \right) dA = -\frac{1}{2} \oint_{\partial A} \psi d(x^2 + y^2)$$

$$= -J_w - \oint_{\partial A} \psi (a dx + b dy) = -J_w - a\psi_y - b\psi_x \tag{15}$$

where:

$$J_w = \iint_A \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] dA \tag{16}$$

$$\oint_{\partial A} \psi dy = \iint_A \frac{\partial \psi}{\partial x} dA = \psi_x \quad \text{and} \quad \oint_{\partial A} \psi dx = -\iint_A \frac{\partial \psi}{\partial y} dA = \psi_y \tag{17}$$

The polar moment of area about the origin, J_p, can be related to J_p^c, the polar moment of cross-section area about point C:

$$J_p = J_p^c + (a^2 + b^2) A \tag{18}$$

The moments of area S_x and S_y in equation (14) can be transformed to those about the axes passing through C correspondingly, following in equation (12). Hence:

$$\begin{aligned} T &= G\alpha\left[-J_w - a\psi_y - b\psi_x + J_p^c + (a^2+b^2)A - a(S_y^c + aA) - b(S_x^c + bA)\right] \\ &= G\alpha\left[J_p^c - J_w - a(S_y^c + \psi_y) - b(S_x^c + \psi_x)\right] \end{aligned} \quad (19)$$

where J_p^c and J_w are independent of a and b. Since the torque is a couple which is independent of the selection of the coordinate system. Thus, one must have:

$$S_x^c + \psi_x = 0 \quad \text{and} \quad S_y^c + \psi_y = 0 \quad (20)$$

The above equations and those in equation (13) together result in:

$$S_x^c = S_y^c = 0 \quad \text{and} \quad \psi_x = \psi_y = 0 \quad (21)$$

Vanishing of the moments of area, S_x^c and S_y^c, reveals that the centre of twist, C, is the very position of the centroid of the cross-section.

In general, the torque is given by

$$T = G\alpha\left(J_p^c - J_w\right) \quad (22)$$

This is not affected by the location of the origin, O.

Discussion and conclusions

It has been proved that the centre of twist coincides with the centroid of the cross-section. The latter is a geometric property of the shape and it can be made known without solving the torsion problem first. This applies to prismatic bars of any cross-section. For example, the centre of twist of a circular tube is obviously the centre of the cross-section. Suppose that a longitudinal slit through the tube is introduced to make it an open section. Although the torsion rigidity of the bar will be reduced significantly, the centre of twist remains where it was because the slit does not alter the location of the centroid of the cross-section. On the contrary, the shear centre moves from the centre of the cross-section of the tube to the outside of the wall on the opposite side of the slit.

The conclusion is valid for bars of multiply connected cross-sections as well as singly connected ones. One of the advantages of the formulation of the torsion problem based on the warping function, i.e. the displacement-based approach, is that it applies to any cross-section, singly or multiply connected, open or closed, without the need for even a slight alteration in the form of governing equation and the boundary condition. (In contrast, in the formulation with the Prandtl stress function, different constants need to be found and applied to the different boundaries.) As a result, the centre of twist coincides with the centroid of the solid part of the cross-section. The centroid may not always be in the solid part, as in the case of a circular tube, but it exists.

The conclusion can also be extended to bars of composite cross-sections, i.e. involving different materials. In this case, the centre of twist will no longer be the geometric centroid of the cross-section but the centroid of the area, weighted by the shear modulus of the material in each part of cross-section.

When a bar is made of a general anisotropic material, the torsion and bending will become coupled in general, in which case the centre of twist needs to be redefined appropriately before the problem is addressed.

Having established the centre of twist, three different ways of formulating the torsion problem with a warping function, which are consistent with the underlying assumptions, can be summarised as follows.

1 If the problem is formulated based on, in addition to the warping, a rigid-body rotation in the plane of the cross-section, the conventional formulation (as given above) is perfect when the origin, O, is chosen at centroid C of the cross-section, i.e. $a = b = 0$.
2 With the same deformation pattern as above, if the origin, O, is located at a different position from C, the formulation in the section 'Torsion problem formulated with an arbitrarily located origin' (above) should be used instead.
3 If the origin, O, is located at a different position from C, yet the governing equation and boundary condition in the conventional formulation (above) is used, the deformation pattern should be described as a rigid-body rotation of the cross-section *in space*, in addition to the warping. Hence the displacement field should be given as follows, instead of equation (1):

$$u = -\alpha(y-b)z$$
$$v = \alpha(x-a)z$$
$$w(x,y) = \alpha(\psi(x,y) - bx + ay) \tag{23}$$

The rigid-body rotation part in the above displacement field is the resultant of a rotation of αa about the x-axis, αb about the y-axis and α about the z-axis.

Since the difference in displacement fields in equations (1) and (23) amounts only to a rigid-body motion, as noticed in for example [6–9], the significance of locating the true centre of twist seems to have been undermined. Indeed, the stress distribution is not affected at all. It has been noted, for example in [14], and can be easily shown, that none of the expression of torque, the governing equation and the boundary condition for the Prandtl stress function is affected by the location of the centre of twist when a stress-based approach is adopted. However, its significance is obvious: it maintains consistency between the formulation of the problem with a warping function and the underlying assumption regarding the deformation pattern.

Having sorted out the consistency as above, formulation (1) is an obvious choice for the torsion problem.

To conclude, the centre of twist for a prismatic bar coincides with the centroid of the cross-section and the conventional formulation is consistent only when the centroid is taken as the origin of the coordinates.

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