On springs and matrices

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Abstract It is shown that the mechanical behavior of an elementary two-spring structure can be used as a direct realization of practically all those concepts of matrix analysis that are usually taught in undergraduate engineering studies: positiveness, symmetry, eigenvalues (vectors) with their orthogonality and extreme properties, invariants, similarity, and so on. The interaction between pure mathematical concepts and a well known physical example creates an associative learning process that helps students to grasp abstract ideas and naturally generalizes scientific concepts from their intuitive level on the physical space to their abstract *n*-dimensional framework. It is therefore proposed that matrix (tensor) algebra should be explored as an inherent part of the basic course in mechanics.

Keywords matrices; mechanics; structures; elastic energy

Introduction and motivation

Many years of personal experience in teaching a large selection of courses in mechanics in the Faculty of Mechanical Engineering at the Technion, Israel, showed that the subject of matrix (tensor) algebra, taught in math courses, and their related concepts, are fundamental tools for today's engineering practice. However, abstract concepts such as eigenvalues and eigenvectors, symmetry, similarity, space, determinants, diagonalization, invariants, and so on, are difficult to comprehend and implement when taught in math courses in a 'sterile' atmosphere. These concepts are usually understood in the most 'technical' sense, without specific relation to real engineering problems, especially when more than three dimensions are involved. Moreover, matrix algebra is usually studied in a 'formal' fashion: 'If such and such conditions are fulfilled, here are the following lemmas and their proofs'. This is 'learning without reason', where almost no association with familiar phenomena is involved. On the other hand, undergraduate-level structure analysis courses are commonly taught as a separate subject, with few references to linear algebra. It is therefore an important teaching challenge in mechanical engineering to develop an interdisciplinary approach, so valuable in a high-tech environment. There are some remarkable examples of this math–physics 'composite' learning [1].

There are many books for advanced reading [2–5, to name a few] that cover math topics as a background for further research in mechanics, but their level of abstraction is too high for engineering undergraduate studies. It is the basic level, combining intuitive physical insight with mathematical structure, which seems to be demanded.

In the following, most of the subject material of the first course in linear algebra, which is so fundamental to all aspects of mechanical, aeronautical and civil engineering, is integrated into the solution of a *single*, simple example in mechanics of structures. It is the purpose of this work to show, by this example, the advantages of this interdisciplinary exposure.

Such an integrated approach lends itself to three teaching formats: a linear algebra course supplemented by 'mechanics of structures' applications; a mechanics course with a matrix (tensor) orientation; and two parallel, well coordinated courses. For practical reasons, only the second method has been successfully implemented in the Technion for the past five years.

Many of the mathematical rules and manipulations in this study are classical, and the same is true for the mechanics part. However, in order to integrate the two parts (which is the main theme here), certain definitions and relations in both subjects had to be briefly reviewed for completeness and reference.

Finally, different symbols for matrices and transformations have been used: [K] for matrix K, K_{ij} and K'_{ij} for elements of K related to $\mathbf{e}_i$ and $\mathbf{e}'_i$ base vectors, and $\mathbf{K}$ for the general, free-base tensor. Index summation convention is used, except for those written in parenthesis.

Basic one-dimensional relations

The mechanical response of a one-dimensional linear spring, loaded at one end (a) and fixed at the other end (b), is characterized by the trivial relation:

$$f = ku, \text{or } u = cf, \quad (k,\ c) > 0 \tag{1}$$

where f is the force, u is the displacement of the spring at (a) from its initial (non-loaded) position and (k, c) are the stiffness and compliance of the spring, respectively. Also,

$$k = c^{-1}, \varepsilon = \left(\frac{1}{2}\right)fu = \left(\frac{1}{2}\right)ku^2 = \left(\frac{1}{2}\right)cf^2 \tag{2}$$

where ε is the elastic energy of the spring. The positive values of k and c ensures that f and u will always have the same sign (direction), from which it is also implied that ε will be always positive for any force or displacement. The force, f, is usually considered the 'action' parameter, u is the 'reaction' (response) of the spring and the linear factor between them is the 'physical property' of the spring under study.

The *same* one-dimensional spring can be examined from a 'two-dimensional perspective'. Specify base directions by two, arbitrarily chosen, orthonormal vectors $\mathbf{e}_1$, $\mathbf{e}_2$. Then, both **f** and **u** have components relative to these base vectors such that:

$$\mathbf{f} = f_i\mathbf{e}_i = f_1\mathbf{e}_1 + f_2\mathbf{e}_2, \quad \mathbf{u} = u_i\mathbf{e}_i = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 \tag{3}$$

$$\mathbf{f} = k\mathbf{u} \tag{4}$$

It is seen that although **f** and **u** have a different physical meaning, their difference is merely a positive factor (or a scalar). Therefore, they *must* represent two quantities (vectors) with *identical* mathematical properties. For example, if the size of **u**, which is the distance that point (a) moved, is always positive and independent of the base vectors ($\mathbf{e}_i$), then the same is true for force **f**. In mathematical language, k

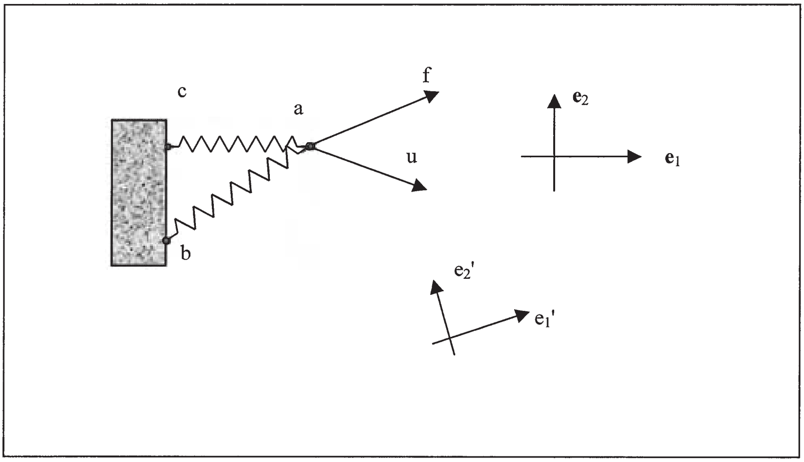

Fig. 1 *A two-spring structure (stiffness k_{ab} and k_{ac}) loaded by a force **f** at (a).*

transforms all points of the two-dimensional space (represented by u_1,u_2) to another two-dimensional space (f_1,f_2). Moreover, if we put the two points on the same two-dimensional plane, this transformation looks 'radial'.

It is seen that an example of a trivial physical phenomena helps students to grasp abstract ideas in mathematics and explains the practical *need* for these ideas.

Basic two-dimensional relations

Consider a combination of two springs, which is the simplest two-dimensional 'structure' (Fig. 1) taught in undergraduate-level mechanics. Apply an external force, **f**, at point (a). The two springs will change their length and point (a) will move to a new position through a displacement vector **u**. A short observation shows that, in this case, the directions of **f** and **u** are *not* necessarily the same. For example, if **f** is parallel to $\mathbf{e}_1$ ($f_2 = 0$), it is very intuitive to see that both $u_1(f_1)$ and $u_2(f_1)$ are nonzero. Moreover, it is not obvious that these two displacements are linear with **f** (only small displacements are considered here), but if they are, then

$$u_1(f_1, f_2 = 0) = C_{11}f_1, \quad u_2(f_1, f_2 = 0) = C_{21}f_1 \tag{5}$$

where C_{ij} are the linear 'two-dimensional spring coefficients', with the first index naturally related to u_i and the second to f_j. By the same argument, if only f_2 is applied, we can write:

$$u_1(f_2, f_1 = 0) = C_{12}f_2, \quad u_2(f_2, f_1 = 0) = C_{22}f_2 \tag{6}$$

If, in addition, the total response $u_1(f_1,f_2)$ is the sum of the separate responses due to f_1 and f_2 (the concept of linearity is conveniently brought up here), then

$$u_1(f_1, f_2) = u_1(f_1) + u_1(f_2) = C_{11}f_1 + C_{12}f_2 \tag{7}$$

Analogously,

$$u_2(f_1,f_2)=u_2(f_1)+u_2(f_2)=C_{21}f_1+C_{22}f_2 \tag{8}$$

and, in a compact form:

$$u_i=C_{ij}f_j \quad \text{or} \quad \mathbf{u}=\mathbf{C}\bullet\mathbf{f} \tag{9a}$$

$$f_i=K_{ij}u_j \quad \text{or} \quad \mathbf{f}=\mathbf{K}\bullet\mathbf{u} \tag{9b}$$

so that:

$$\mathbf{u}=\mathbf{C}\bullet(\mathbf{K}\bullet\mathbf{u})=\mathbf{I}\bullet\mathbf{u}\rightarrow\mathbf{C}\bullet\mathbf{K}=\mathbf{I},\quad C_{ij}K_{jm}=\delta_{im} \tag{10}$$

where the dot (•) symbol represents vector inner product (one index summation). We have also,

$$\mathbf{C}=\mathbf{K}^{-1},\quad C_{ij}=\left(\mathbf{K}^{-1}\right)_{ij} \tag{11}$$

The physical meaning of elements C_{ij} is simple: the displacement in direction $\mathbf{e}_i$ due to a unit force in direction $\mathbf{e}_j$. Note that there is no need for 'naming' a specific index by 'row' and the other by 'line' and so on. Moreover, it is seen that, contrary to the abstract exposition process in a math course, the matrix definition and its basic 'operations' on a vector (equations 7–11) are obtained as a result of a practical *need* to solve a physical problem.

Transformation of base vectors

The components of **f** are different when related to different orthonormal base vectors, but the force itself is the same in any system. That is:

$$\mathbf{f}=f_i\mathbf{e}_j=f_i'\mathbf{e}_i' \tag{12a}$$

so that

$$f_i=\mathbf{f}\bullet\mathbf{e}_i=f_j'\mathbf{e}_j'\bullet\mathbf{e}_j=a_{ij}f_j' \tag{12b}$$

where [a] is a matrix defined by:

$$a_{ij}=\mathbf{e}_i\bullet\mathbf{e}_j' \tag{13}$$

a can be interpreted as transforming the components f_i to f_j'. Since both f_i and f_j' have the same size, **a** describes a 'rotation'. Moreover, comparing the size of **f** in the two systems yields:

$$f_if_i=a_{ij}f_j'a_{im}f_m'=f_m'f_m'=\delta_{mj}f_m'f_j' \tag{14}$$

from which the unique property of a_{ij} is drawn:

$$a_{ij}a_{im}=\delta_{mj}$$

or, in vector notation:

$$\mathbf{a}\bullet\mathbf{a}^T=\mathbf{I} \tag{15}$$

As for **f** and **u**, the specific values of the components of **C** (or **K**) depend on the choice of the base vectors. It is therefore clear that there must be some relation between any two sets of components (related to the two base vectors) of **C**. Moreover, we suspect that, if the size of a vector must be the same in any base vector, some functions of the components of **C** should have the same values for all 'realizations', too. These functions are termed 'invariants'. Let us write the force in terms of two base vectors, $\mathbf{e}_i$ and $\mathbf{e}_i'$. Using equations (13–15),

$$f_i = K_{ij}u_j = a_{ij}f_j' = a_{ij}K_{jm}'u_m' = a_{ij}K_{jm}'a_{km}u_k \tag{16}$$

Since equation (16) is valid for *any* **u**,

$$K_{ik} = a_{ij}K_{jm}'a_{km}$$

or, in matrix notation:

$$[K] = [a][K'][a]^T \tag{17}$$

which is the formal mathematical relation between two 'similar' matrices. It is seen that the similarity condition is a reflection of a *simple* idea that both matrices represent the property of the *same* physical entity (stiffness), but relative to different systems of coordinates. This observation is closely related to the more general and profound idea of 'objectivity', which is so fundamental in physics.

It is seen that abstract ideas in matrix analysis can be easily drawn from simple, intuitive, physical behavior. Therefore, teaching the two aspects as a 'composite' unit has benefits in terms of students' understanding in both areas. We now approach some more advanced topics.

Positive definite matrix

Since the total elastic (internal) energy of the structure, which is equal to the work done by the external force, is composed of the sum of energies of all springs, and each spring has a positive elastic energy, the total external work done on the structure is also positive. Therefore,

$$2\varepsilon = \mathbf{f} \bullet \mathbf{u} = (\mathbf{K} \bullet \mathbf{u}) \bullet \mathbf{u} = (\mathbf{C} \bullet \mathbf{f}) \bullet \mathbf{f} > 0 \tag{18}$$

This feature is true for *any* displacement (or force), so it is a 'net' property of **K** and **C** (structure), termed mathematically a 'positive definite matrix'. Moreover, from equation (18) it is seen that the angle between **f** and **u** must be smaller than $\pi/2$. This is equivalent to saying that **f** must have a positive component in the direction of **u** and vice versa. It is a natural two-dimensional generalization of the 'positivness' condition ($k > 0$) in the one-dimensional case, which is expanded to multidimensional problems with no effort.

When choosing **u** in the $\mathbf{e}_1$ direction, equation (18) shows that K_{11} must be positive, and the same is true for $\mathbf{e}_2$ and K_{22}. Since the values of K_{ij} are independent of

f, the tensor positive definiteness induces also positive 'normal' (diagonal) components of matrices [K] and [C] in *any* coordinate system.

Symmetry

It is clear that the matrix **C** (or **K**) is the analogous two-dimensional spring compliance (or stiffness) of the one-dimensional case, which brings up many 'analogous' questions to ones that are trivial for the one-dimensional case, such as: How many elements of **C** are independent? From a physical perspective, given two springs with stiffness k_{ab} and k_{ac}, one can connect the two (a) edges, and fix the end (b) to the wall, as in Fig. 1. Then, by fixing point (c) (or some angle α between ab and ac), the structure is completely defined. Therefore, only three independent parameters (k_{ab}, k_{ac}, α) are involved and it is clear that the four elements of **C** are somehow related.

To find the specific relation, consider the elastic energy of each spring, which is a function of its final extension only, and is independent of the particular loading path. Clearly, this path independence must hold for the total energy of the structure, too (energy is additive). To check the consequence of this property, load the structure successively by two forces, $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$, for which $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$ are the reactions when loaded separately. Then, for simultaneous loading, verify that the total energy is the same for the two possibilities of loading sequences. If $\mathbf{f}^{(1)}$ is applied first, the total energy of the structure is:

$$\varepsilon^{(1)} = \left(\frac{1}{2}\right)\mathbf{f}^{(1)} \bullet \mathbf{u}^{(1)} + \left(\frac{1}{2}\right)\mathbf{f}^{(2)} \bullet \mathbf{u}^{(2)} + \mathbf{f}^{(1)} \bullet \mathbf{u}^{(2)} \tag{19}$$

where the last term involves the work done by $\mathbf{f}^{(1)}$ while $\mathbf{f}^{(2)}$ is already applied. Similarly, if $\mathbf{f}^{(2)}$ is applied first,

$$\varepsilon^{(2)} = \left(\frac{1}{2}\right)\mathbf{f}^{(2)} \bullet \mathbf{u}^{(2)} + \left(\frac{1}{2}\right)\mathbf{f}^{(1)} \bullet \mathbf{u}^{(1)} + \mathbf{f}^{(2)} \bullet \mathbf{u}^{(1)} \tag{20}$$

That is,

$$\mathbf{f}^{(1)} \bullet \mathbf{u}^{(2)} = \mathbf{f}^{(2)} \bullet \mathbf{u}^{(1)}$$

or $\quad \left(\mathbf{K}\mathbf{u}^{(1)}\right) \bullet \mathbf{u}^{(2)} = \left(\mathbf{K}\mathbf{u}^{(2)}\right) \bullet \mathbf{u}^{(1)}$

or $\quad K_{ij} = K_{ji} \qquad (21)$

which is the mathematical condition for symmetry of the matrix **K** (or **C**) and shows that indeed only three components of **C** are independent. Furthermore, it shows that the displacement in $\mathbf{e}_i$ due to a unit force in $\mathbf{e}_j$ equals the displacement in $\mathbf{e}_j$ due to a unit force in $\mathbf{e}_i$. This is a 'small-scale' variant of the famous Maxwell reciprocal theorem, which is used extensively in mechanics. The above 'path independence' property can be further explored to demonstrate other, more advanced relations (such as Gauss theorems), but this will not be given here.

Note also that all the above can be expanded to three springs in three dimensions, without *any* modification.

Principal directions and eigenvalues

If $\mathbf{f}$ is *not* necessarily parallel to $\mathbf{u}$, it is interesting to examine whether there are any two *specific* 'action–reaction' vectors, $\mathbf{f} = \mathbf{f}^S$, and $\mathbf{u} = \mathbf{u}^S$, which are parallel. This is an important practical question, since in many applications it is desirable that the reaction will not have any trace in the direction perpendicular to the direction of the action vector. For example, if a certain part of a machine is deformed under loading, it is important that this deformation will not interfere with neighboring parts which are in contact. Therefore, it is necessary to find $\mathbf{f}^S$ such that

$$\mathbf{u}^S = c\mathbf{f}^S = c\mathbf{I}\mathbf{f}^S = \mathbf{C}\mathbf{f}^S \tag{22}$$

where c is the spring 'eigen' compliance and $\mathbf{u}^S$, $\mathbf{f}^S$ are 'eigen' displacements and forces. Therefore,

$$\mathbf{C} \bullet \mathbf{f}^S - c\mathbf{I} \bullet \mathbf{f}^S = (\mathbf{C} - c\mathbf{I}) \bullet \mathbf{f}^S = 0 \tag{23}$$

from which one can find the two eigenvalues $c^{(1)}$, $c^{(2)}$ and the two corresponding eigenforces $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$ or eigen displacements.

It is obvious, both by intuition and by equation (23), that if $\mathbf{f}^S$ is an eigenforce, any multiplication of $\mathbf{f}^S$ by a scalar is an eigenforce too. Therefore, the most important information here is the *direction* of the force, not its size. That is,

$$\mathbf{n}^S = n^S{}_1\mathbf{e}_1 + n^S{}_2\mathbf{e}_2 = \mathbf{f}^S / |\mathbf{f}^S| = \mathbf{u}^S / |\mathbf{u}^S| \tag{24}$$

where the components $n^S{}_i$ are the cosine values of the angles between $\mathbf{n}^S$ and $\mathbf{e}_i$. We see that only when we normalize the eigenvectors by their size do their elements have a real physical meaning. This reason is usually ignored in math courses.

Angle between principal directions

As stated above, the symmetry of [K] means that a displacement in direction $\mathbf{e}_1$ due to a unit force in direction $\mathbf{e}_2$ (i.e., $u_1(f_2 = 1, f_1 = 0)$) is equal to $u_2(f_1 = 1, f_2 = 0)$. This is true for [K] in any orthonormal base. Therefore, if we choose $\mathbf{e}_1$ parallel to $\mathbf{n}^{(1)}$, then $\mathbf{n}^{(2)}$ *must* be parallel to $\mathbf{e}_2$, which means that $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ are orthogonal. This intuitive physical observation is not so easily proved mathematically, although the algebraic part of the proof is straightforward. Let

$$\mathbf{K} \bullet \mathbf{n}^{(1)} = k^{(1)}\mathbf{n}^{(1)}, \quad \mathbf{K} \bullet \mathbf{n}^{(2)} = k^{(2)} \bullet \mathbf{n}^{(2)} \tag{25}$$

Cross-scalar multiplication yields:

$$k^{(2)}\left(\mathbf{K} \bullet \mathbf{n}^{(1)}\right) \bullet n^{(2)} = k^{(1)}\left(\mathbf{K} \bullet \mathbf{n}^{(2)} \bullet \mathbf{n}^{(1)}\right) = k^{(1)}k^{(2)}\mathbf{n}^{(2)} \bullet \mathbf{n}^{(1)} \tag{26}$$

Combining equations (25) and (26), it is seen that unless $k^{(1)} = k^{(2)}$ (a particular case which will be discussed later), $\mathbf{n}^{(2)} \bullet \mathbf{n}^{(1)} = 0$.

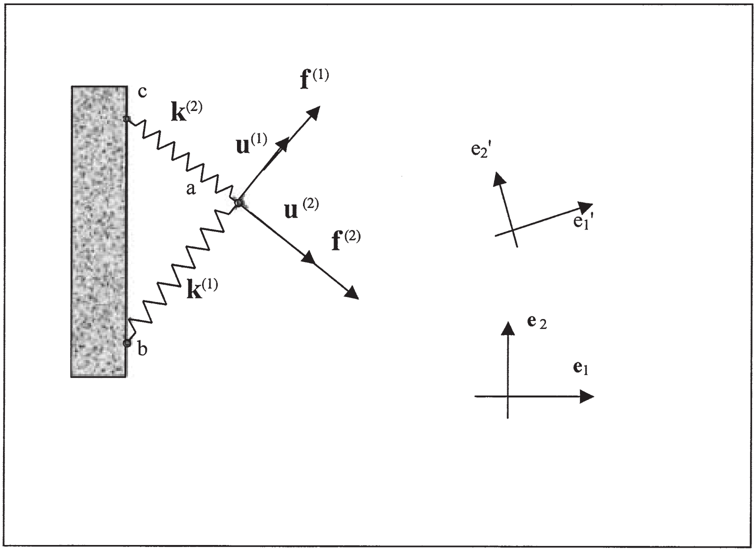

Fig. 2 *Equivalent two-dimensional spring with two 'eigen' springs in the principal directions.*

A vivid physical meaning of the above can be described by the following: for each two-dimensional structure such as that shown in Fig. 1, there is an equivalent structure such as that shown in Fig. 2, for which the stiffness **K** is identical.

The two principal equivalent springs are perpendicular to each other, with principal stiffness $k^{(n)}$. It is now obvious why a force in one principal direction causes no displacement in the other direction. Moreover, the eigenvalues $k^{(i)}$ have a simple realization as the stiffness of these 'eigen' springs.

Examining the equivalent principal structure, another important observation is easily made. By definition, $K_{nn}(\mathbf{n})$ is the ratio (stiffness) between a force in direction **n** and the component of **u** in **n**. From the *physical* symmetry of the structure, it is obvious that K_{nn} is symmetric relative to both $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$; that is, K_{nn} will have the *same* value for two forces having opposite angles relative to $\mathbf{n}^{(1)}$ or $\mathbf{n}^{(2)}$. Therefore, $K_{nn}(\mathbf{n} = \mathbf{n}^{(1)}$ or $\mathbf{n}^{(2)})$ must possess two *extremum* values of K_{nn}. Since for the two-dimensional case only two principal springs are possible, one of them *must* be the maximum and the other the minimum. Again, the formal mathematical proof (involving in the general case the Lagrange multipliers method) is not straightforward, at least for the undergraduate engineering level.

Consider now a very special two-dimensional structure, which involves two principal springs of the same magnitude, that is, $k^{(n)} = k$. Since $k^{(n)}$ are the extreme values of all K_{nn} in any direction, it is trivial to observe that $K_{nn} = k$ in *any* direction, and the matrix [K] will be equal to $k\mathbf{I}$ ($K_{ij} = k\delta_{ij}$) for *any* rotated base vectors $\mathbf{e}_i'$. The state of stress (pressure) in a fluid in static equilibrium is a good example

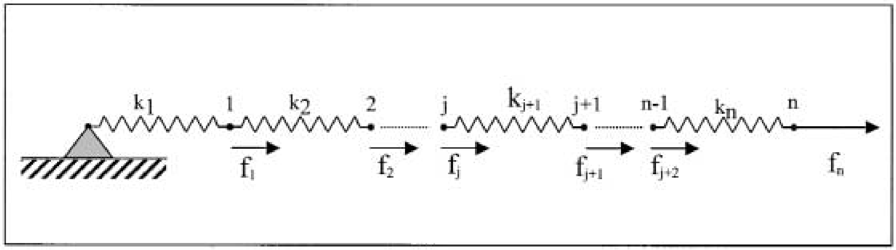

Fig. 3 *A chain of springs which is one dimensional on the physical space and n dimensional on the mathematical abstract space.*

for this type of matrix. Therefore, it is sometimes termed a 'hydrostatic' matrix (structure).

Dimension, space and a tridiagonal matrix

Consider a natural expansion to the one-dimensional problem of a single spring discussed under 'Basic one-dimensional relations', above, as seen in Fig. 3. There are n springs of various stiffness connected as one chain and loaded by n forces at the junction points. Similar to the single-spring case, we like to find the relation between the forces and the displacements at the corresponding points. From elementary 'statics' course, equilibrium can be applied by either the global or the local method.

The global method. Make a virtual 'cut' at a certain section (say, between J and J + 1) and apply equilibrium on the whole section between the cut and the right end such that:

$$F = k_j\left(u_{j+1} - u_j\right) = f_{j+1} + f_{j+2} + \ldots f_n \tag{27}$$

Which yields a set of equations:

$$k_n(u_n - u_{n-1}) = f_n \tag{28}$$

$$k_{n-1}(u_{n-1} - u_{n-2}) = f_n + f_{n-1} \tag{29}$$

$$k_{n-2}(u_{n-2} - u_{n-3}) = f_n + f_{n-1} + f_{n-2} \quad \text{etc.} \tag{30}$$

$$k_1(u_1) = f_n + f_{n-1} + f_{n-2} + \ldots f_1 \tag{31}$$

The local method. Apply equilibrium on each joint, that is,

$$k_n(u_n - u_{n-1}) = f_n \tag{32}$$

$$k_n(u_n - u_{n-1}) - k_{n-1}(u_{n-1} - u_{n-2}) = f_{n-1} \tag{33}$$

$$k_{n-1}(u_{n-1} - u_{n-2}) - k_{n-2}(u_{n-2} - u_{n-3}) = f_{n-2} \qquad \text{etc.} \tag{34}$$

$$k_2(u_2 - u_1) - k_1(u_1) = f_1 \tag{35}$$

We see that the local method leads directly to the following matrix form:

$$\begin{Bmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{Bmatrix} = \begin{bmatrix} -(k_1+k_2) & k_2 & 0 & 0 \\ k_1 & -(k_1+k_2) & k_2 & \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -k_n & k_n \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{Bmatrix} \tag{36}$$

which is a direct reflection of equations (32–35). Now, notice that the only nonzero elements of **K** are those which are on its diagonal or on adjacent rows on both sides. This is the very well known 'tridiagonal' matrix, which is regularly studied in the first course on numerical analysis. Numerically, it is a unique matrix, since the number of operations (substitutions) needed to solve these n equations is n, instead of n! (Gauss elimination method). However, when we do the n substitutions in equation (36), we get exactly the 'global' equilibrium equations from equations (28–31)! These can be solved consecutively, starting from equation (31) and proceeding backwards. This is a good example of how a numerical solution procedure, studied in a purely mathematical environment, is a reflection of simple 'mechanical' transformation from a local to a global equilibrium.

Another interesting subject which is convenient for discussion here is related to dimensions. The above chain is, physically, a one-dimensional problem, since all displacements and forces have the same direction. However, from the mathematical view, this is an n-dimensional problem. Mathematically, there is only one force **f** and one **u**, having n elements each, but these vectors cannot be described simply on the 'physical' space, but on a more abstract n-dimensional space. If this is so, how do we define the 'size' of these vectors? The simplest guess is:

$$|\mathbf{f}| = (f_i f_i)^{\frac{1}{2}} \tag{37}$$

which is just a 'copy' from the two-dimensional case, and the same goes for **u**. And how can we define an *angle* (α) between two forces **f** and **g**? Again, the simplest guess is:

$$\cos(\alpha) = \frac{f_i g_i}{|\mathbf{f}| \cdot |\mathbf{g}|} \tag{38}$$

Equations (37) and (38) are fundamental in the studying of many subjects in mathematics, starting from groups, linear algebra and even functional analysis. The opportunity given by this simple example to expand our knowledge (and stimulate curiosity) to general, more abstract multidimensional cases is clear.

There are additional mechano-math issues which can be further explored here, such as the transformation of the n discrete equations into a continuous differential equation, but they will not be discussed further.

Summary and Conclusions

Teaching mathematics as part of the process of exploring the laws of physics (basic mechanics of materials in this study) is characterized by a natural *research-oriented* process: identifying simple relations (laws) in a one-dimensional system; expansion to a two-dimensional space; checking that the two-dimensional laws are valid for the three-dimensional case; and generalizing to a fully abstract n-dimensional space. In these 'generalizing' processes, the 'guessing and checking' stage, which is so fundamental in research, is strongly emphasized.

Important conclusions from this 'inductive' process, as seen from the above study, are:

- 'Scaling up' by physical examples (springs) is an effective way for understanding major concepts, which are usually difficult to comprehend in a purely mathematical, isolated environment.
- The learning process is intuitive and self-explanatory, up to the three-dimensional case, and includes all mathematical 'features' and formal techniques.
- Generalization to n dimensions needs no further mathematical tools. Therefore, scaling up to a higher dimensional analysis is a matter of abstraction only, and teaching at that stage can be concentrated solely on that abstraction and the generalized interpretations.
- The potential of exploring so many aspects of matrix analysis (usually covered in two semesters) by such a simple structure (three springs are enough) should be noticed. It is a strong sign that there are simple examples in *other* fields that can be used alternatively, and be brought from the private experience of the teacher. Examples from more than one field are the ultimate proof of the unified language of mathematics, and appreciating this property is one of the major goals of teaching science and engineering.

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