
Optimising an intercooled compressor for an ideal gas model

Jeffery D. Lewins

Magdalene College, University of Cambridge, Cambridge CB3 0AG, UK

E-mail: jdl@eng.cam.ac.uk

Abstract Many of the conventional results obtained when optimising the performance of an intercooler during compression using a perfect gas model can be obtained when the restrictions of the model are relaxed to an ideal gas. That is, we now have temperature-dependent specific heat capacities but retain the equation of state $pV = RT$. This note illustrates the theme.

Key words ideal gas; perfect gas; intercooler; compression; optimisation

Working in the perfect gas model, in which $pV = RT$ and $c_p = \text{const}$, leads to some elementary results for an optimisation of a gas turbine that gives maximum work per unit mass circulated. These are useful for giving a feel to more realistic results in gas turbine and Joule cycle theory. It has been recently shown that the essence of these results remains true, albeit with a different emphasis, when the model is relaxed to the ideal gas, $pV = RT$ only. Those results have been given for a Joule cycle optimised for maximum specific work [1, 2].

In this note we show the analogous development for a two-stage intercooled compressor. Minimisation of the specific work input, for an ideal compressor with 100% isentropic efficiency and 100% heat exchanger effectiveness, is readily seen to require that the stage pressure ratio rise should be the square root of the overall pressure ratio in the more restrictive perfect gas model. It is not always pointed out what the consequences are for the temperatures at the two output stages: the stage outlet temperatures are equal. When the model is relaxed to that of an ideal gas, we show here that the temperature result is still valid in this more general model. This result is then generalised further by consideration of pressure losses in the heat exchanger, isentropic efficiencies less than 100% in the compressor stages, and limitations in the heat exchanger effectiveness. Indeed, the results carry over to multi-stage intercooling. In the following section we give the conventional perfect gas theory and in the subsequent section the generalisation to an ideal gas.

Perfect gas

Fig. 1 shows the components of a two-stage compressor with a single intercooler. To start with, we suppose that there are no pressure losses in the heat exchanger, perfect isentropic efficiencies in both adiabatic stages of compression and a perfect heat exchanger with effectiveness of unity. The last of these assumptions means that the first-stage compressed gas is returned to the ambient temperature it started at, so that the second stage may be taken as starting from ambient (Fig. 2).

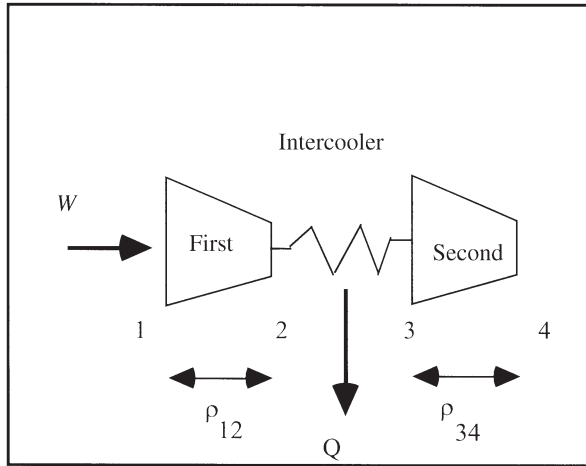


Fig. 1. Components of the two-stage intercooled compressor.

Using the second master equation of thermodynamics, $dh = Tds + vdp = c_p dT$ for the ideal gas, and c_p constant for the perfect gas, we have the usual result that the temperature rise in a reversible adiabatic compression through a pressure ratio r is given by

$$\frac{T}{T_0} = (r)^{R/c_p} = r^{\frac{\gamma-1}{\gamma}} \equiv \rho \tag{1}$$

(where ρ is *not* the density but as defined). The work of compression per unit of mass flow over the two stages is given by:

$$\begin{aligned} W &= \int_1^2 v dp + \int_3^4 v dp = c_p [T_2 - T_1 + T_4 - T_3] = c_p T_1 [\rho_{12} + \rho_{34} - 2] \\ &= c_p T_1 \left[\rho_{12} + \frac{\rho}{\rho_{12}} - 2 \right] \end{aligned} \tag{2}$$

To minimise this over the intermediate pressure ratio, we differentiate with respect to ρ_{12} and, as a necessary condition for an optimum, equate the result to zero to find that the optimum intermediate pressure ratio is $\rho_{12} = \sqrt{\rho}$ and $r_{12} = r_{34}$. As a consequence, not only does $T_1 = T_3$ by virtue of the perfect heat exchanger, but $T_2 = T_4$ at a minimum compressor work input, as claimed in the introductory section.

These results may now be generalised within the perfect gas model. First, consider the effect of a pressure loss in the heat exchanger by $\rho_0 < 1$ so that between them the two stages must generate ρ/ρ_0 . Fig. 3 shows the intermediate drop in pressure with unchanged heat exchanger exit temperature. The additional compression is then to be shared equally between the two stages, so that for each stage $\rho_{stage,opt} = \sqrt{\rho/\rho_0}$ and again the stage exit temperatures are equal.

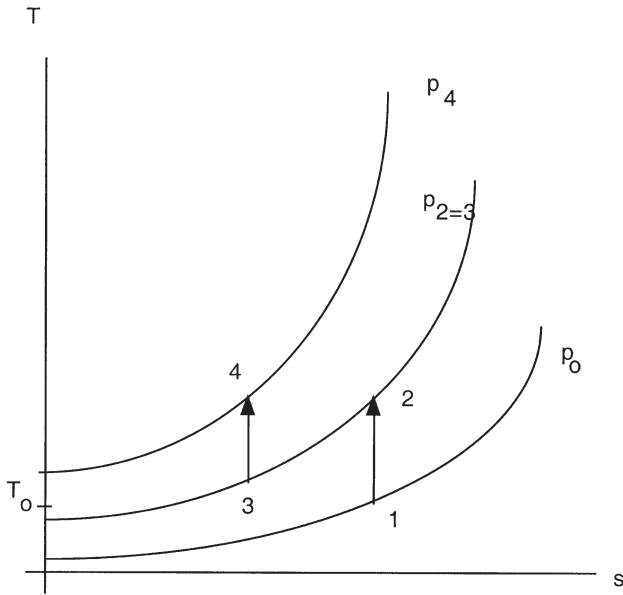


Fig. 2. The $T-s$ diagram for the two-stage intercooled compressor.

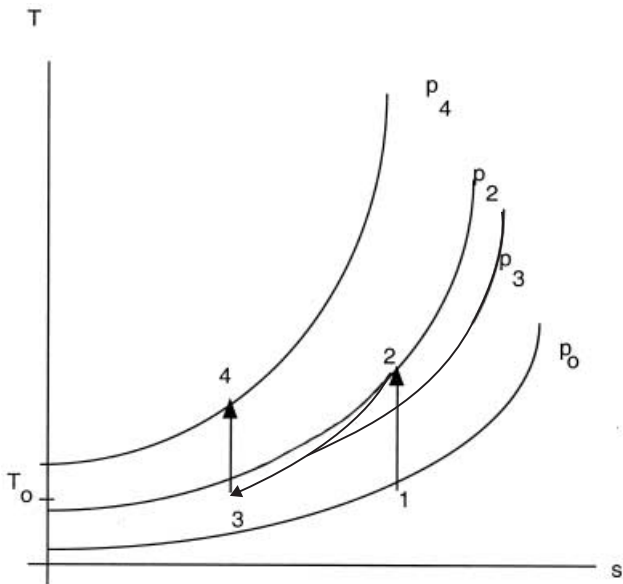


Fig. 3. The $T-s$ diagram with pressure loss in the intercooler.

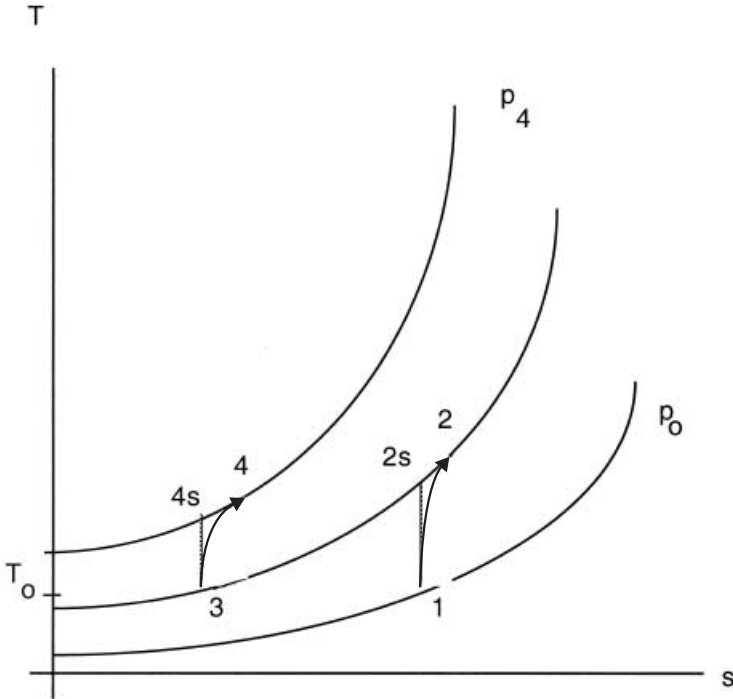


Fig. 4. The T-s diagram with compressor inefficiencies.

Next, consider adiabatic but irreversible compression (Fig. 4), described by a common compressor isentropic efficiency [3]:

$$\eta = \frac{T_{2s} - T_1}{T_2 - T_1} = \frac{T_{4s} - T_{3=1}}{T_4 - T_{3=1}} \tag{3}$$

where the lower temperatures are the same for the perfect heat exchanger. We would find that the optimum design is still to share the pressure ratio equally between the stages. Although stage exit temperatures are higher, the rise is the same for both stages and the exit temperatures remain equal.

Then allow [4] for a heat exchanger effectiveness $\epsilon < 1$, where

$$\epsilon = \frac{T_2 - T_3}{T_2 - T_1} \tag{4}$$

so that the exchanger exit temperature is not lowered to ambient (Fig. 5). With all these effects, the non-dimensional compressor work is:

$$\begin{aligned} W^* &\equiv \frac{W}{c_p T_1} = \left[\frac{T_4}{T_1} - \frac{T_3}{T_1} + \frac{T_2}{T_1} - 1 \right] \\ &= \left[(1 - \epsilon) \left(1 + \frac{\rho_{12} - 1}{\eta} \right) + \epsilon \right] \left[\frac{\rho / \rho_0 \rho_{12} - 1}{\eta} + \frac{\rho_{12} - 1}{\eta} \right] \end{aligned} \tag{5}$$

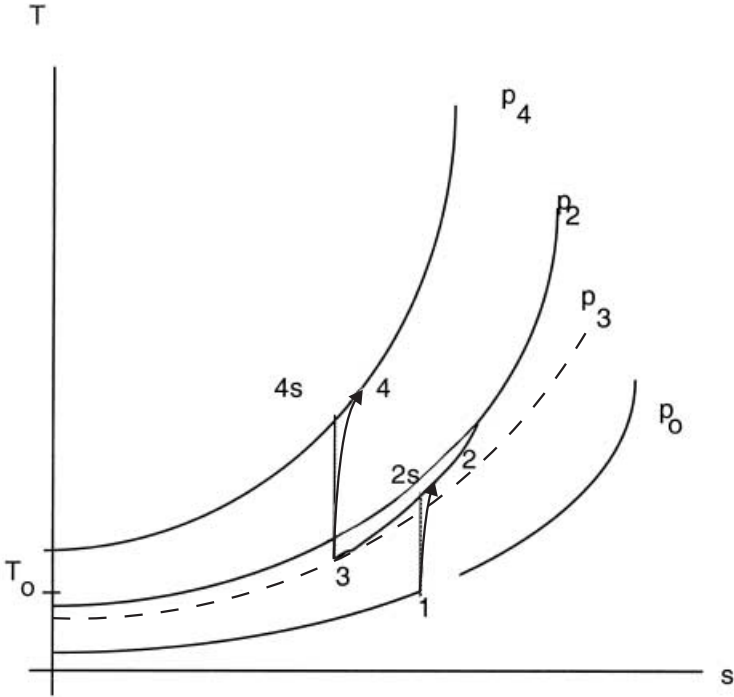


Fig. 5. The T - s diagram with pressure loss, isentropic inefficiencies and an imperfect heat exchanger.

which on differentiation and equating to zero yields the simple result that:

$$\rho_{12,opt} = \sqrt{\rho/\rho_0} = \rho_{34,opt} \tag{6}$$

With the heat exchanger ineffectiveness allowed for, it is no longer true that the stage exit temperatures are the same at optimisation. In fact,

$$\frac{T_4}{T_2} = \frac{T_3}{T_1} = \varepsilon + (1 - \varepsilon) \left(1 + \frac{\sqrt{\rho/\rho_0} - 1}{\eta} \right) \tag{7}$$

If the effectiveness is unity then this reduces to:

$$\frac{T_4}{T_2} = \frac{T_{4s}}{T_{2s}} = \frac{T_3}{T_1} = 1 \tag{8}$$

independent of the compressor efficiency. The detail of these results is given in Table 1.

Finally, all these results may be generalised to multi-stage intercooler systems by reference to Bellman’s dictum [5], that any part of an optimum path must itself be optimum. Thus the equal sharing of the compression ratio between stages is the condition for a minimum compressor work with all the above losses taken into account.

The result – that the pressure rise is shared equally between the two stages and the outlet temperatures are equal – may be readily visualised from Fig. 2. Due to

TABLE 1 *Optimum values for the two-stage perfect gas intercooled compression*

W_{opt}^*	$\frac{\sqrt{\rho/\rho_0}-1}{\eta} \left[2 + \frac{1-\varepsilon}{\eta} (\sqrt{\rho/\rho_0}-1) \right]$
W_{opt}/W 1 stage	$\frac{\sqrt{\rho/\rho_0}-1}{\rho-1} \left[2 + \frac{1-\varepsilon}{\eta} (\sqrt{\rho/\rho_0}-1) \right]$
T_2/T_1 opt	$1 + \frac{\sqrt{\rho/\rho_0}-1}{\eta}$
T_3/T_1 opt	$\varepsilon + (1-\varepsilon) \left(1 + \frac{\sqrt{\rho/\rho_0}-1}{\eta} \right)$
T_4/T_1 opt	$\left[\varepsilon + (1-\varepsilon) \left(1 + \frac{\sqrt{\rho/\rho_0}-1}{\eta} \right) \right] / \left[1 + \frac{\sqrt{\rho/\rho_0}-1}{\eta} \right]$
$T_4/T_2 = T_3/T_1$ opt	$\varepsilon + (1-\varepsilon) \left(1 + \frac{\sqrt{\rho/\rho_0}-1}{\eta} \right)$
T_4/T_1 1 stage	$1 + \frac{\rho-1}{\eta}$

the rising exponentials taken by the isobars, the later parts of an isentropic pressure rise require more work than the initial part. It is thus advantageous to cool the first stage as far as possible so that the second part of the pressure rise starts where the enthalpy rise and hence work input is returned to its smallest value. If one stage has a higher pressure rise than the other, then that stage will do not only more work but unnecessary work; the rise should be shared until both stages make equal use of the low-work initial pressure rise. This remains true, in the perfect gas model, even if (both) the stage efficiencies are less than unity and even if the heat exchanger effectiveness is less than unity; loading the second stage back on the first stage would increase the first-stage outlet temperatures and hence raise the second-stage inlet temperature to more than offset any gain in overall work decrease. This last result, however, is not generally true in the ideal gas model.

Ideal gas model

To extend this analysis to an ideal gas means dropping the simple result for the temperature ratio in perfect adiabatic compression in terms of the pressure ratio. In its place, consider the work per unit mass passed through the 100% isentropic efficient compressor as

$$W = \int_1^2 dh + \int_3^4 dh = \int_1^2 c_p(T) dT + \int_3^4 c_p(T) dT \quad (9)$$

The pressure ratio is given by the more general form

$$R \ln(r) = \int_1^2 \frac{c_p(T)}{T} dT + \int_{3=1}^4 \frac{c_p(T)}{T} dT \tag{10}$$

That is, the second relation provides a constant against which the first relation is to be minimised over two variables, T_2 and T_4 . It is then convenient to use Lagrange multipliers for this constrained optimisation problem [6] and write the Lagrangian as:

$$L \equiv W + \lambda \left[R \ln(r) - \int_1^2 \frac{c_p(T)}{T} dT - \int_1^4 \frac{c_p(T)}{T} dT \right] \tag{11}$$

The problem may now be treated as freely optimised over both variables and the minimum requires:

$$\begin{aligned} \frac{\partial L}{\partial T_2} = 0 &= c_p(T_2) - \lambda \frac{c_p(T_2)}{T_2} \\ \frac{\partial L}{\partial T_4} = 0 &= c_p(T_4) - \lambda \frac{c_p(T_4)}{T_4} \end{aligned} \tag{12}$$

from which it follows that at the minimum $T_2 = T_4$. It also follows that $T_{2s} = T_{4s}$. Thus, what for the perfect gas was a corollary of the optimisation theorem, becomes the chief result for an ideal gas (see Table 2). We also see that small changes of the required pressure ratio lead to small changes in the minimum work given by the Lagrange multiplier:

$$\lambda = \frac{dW_{opt}}{dR \ln(r)} = T_2 = T_4 = \frac{r}{R} \frac{dW_{opt}}{dr} \tag{13}$$

The same generalisations may be made for compressor efficiency and heat exchanger effectiveness. There is a choice of whether to write these as before, in terms of temperatures, or the more general enthalpies. The latter seem to give a ‘cleaner’ result although the practical difference is small¹. To do this we introduce further Lagrange multipliers and distinguish between the efficiency of the first- and second-stage compression. Thus we write the objective function as:

$$W \equiv L_0 = h_2 - h_1 + h_4 - h_3 \tag{14}$$

together with constraints and their Lagrange multipliers

¹ It is certainly customary to use the enthalpy form for isentropic efficiencies. For the heat exchanger, driven by temperature differences, it is a mute point whether to think in terms of the driving forces in temperatures or the heat exchanged in enthalpies. It has to be recollected that the value of either efficiency or effectiveness is measured or calculated for particular operating conditions. The hope is that if these conditions are changed slightly, the value remains unchanged. In the ideal gas model, this will not be exactly true, but there is no *a priori* way of saying one form of the ‘constant’ is more constant than the other. Both forms will be the same in the perfect gas model, of course.

TABLE 2 *Perfect gas Lagrange multipliers*

λ	$r \frac{dW_{opt}^*}{dRdr}$	$\frac{T_{4s}}{\eta_{34}}$
χ	$\frac{dW_{opt}^*}{d\eta_{12}}$	$-\frac{h_2 - h_1}{\eta_{34}} \frac{T_{4s}}{T_{2s}}$
κ	$\frac{dW_{opt}^*}{d\eta_{34}}$	$-\frac{h_4 - h_3}{\eta_{34}}$
μ	$\frac{dW_{opt}^*}{d\varepsilon}$	$-(h_2 - h_1) \frac{T_{4s} - T_3}{\eta_{34} T_3}$

$$L_a \equiv \lambda \left[R \ln(r) - \int_{T_1}^{T_2} \frac{dh}{T} - \int_{T_1}^{T_3} \frac{dh}{T} \right] \quad (15a)$$

$$L_b \equiv \chi \left[\eta_{12} - \frac{h_{2s} - h_1}{h_2 - h_1} \right] \quad (15b)$$

$$L_c \equiv \mu \left[\varepsilon - \frac{h_2 - h_3}{h_2 - h_1} \right] = \mu \left[\varepsilon - 1 + \frac{h_3 - h_1}{h_2 - h_1} \right] \quad (15c)$$

$$L_d \equiv \kappa \left[\eta_{34} - \frac{h_{4s} - h_3}{h_4 - h_3} \right] = \kappa \left[\eta_{34} - 1 + \frac{h_4 - h_{4s}}{h_4 - h_3} \right] \quad (15d)$$

Then define the Lagrangian, a function of five now independent variables, as

$$L \equiv L_0 + L_a + L_b + L_c + L_d \quad (16)$$

and make the five-dimensional gradient $\nabla L = 0$ as a necessary condition at a minimum. This provides:

$$\begin{aligned} 0 &= \frac{\partial L}{\partial T_2} = c_p(T_2) \left[1 + \chi \frac{h_{2s} - h_1}{(h_2 - h_1)^2} - \mu \frac{h_3 - h_1}{(h_2 - h_1)^2} \right] \\ &= c_p(T_2) \left[1 + \chi \frac{\eta_{12}}{h_2 - h_1} - \mu \frac{1 - \varepsilon}{h_2 - h_1} \right] \end{aligned} \quad (17a)$$

$$0 = \frac{\partial L}{\partial T_{2s}} = -\lambda \frac{c_p(T_{2s})}{T_{2s}} - \chi \frac{c_p(T_{2s})}{h_2 - h_1} \quad (17b)$$

$$0 = \frac{\partial L}{\partial T_3} = -c_p(T_3) + \lambda \frac{c_p(T_3)}{T_3} + \mu \frac{c_p(T_3)}{h_2 - h_1} + \kappa \frac{1 - \eta_{34}}{h_4 - h_3} c_p(T_3) \quad (17c)$$

$$0 = \frac{\partial L}{\partial T_4} = c_p(T_4) \left[1 + \kappa \frac{h_{4s} - h_3}{(h_4 - h_3)^2} \right] = c_p(T_4) \left[1 + \kappa \frac{\eta_{34}}{h_4 - h_3} \right] \quad (17d)$$

$$0 = \frac{\partial L}{\partial T_{4s}} = -\lambda \frac{c_p(T_{4s})}{T_{4s}} - \kappa \frac{c_p(T_{4s})}{h_4 - h_3} \quad (17e)$$

The additional advantage of Lagrange multipliers is the information they provide on a small change of *constraint* on the objective function, while keeping the system optimised. Thus we obtain

$$\kappa = \frac{dW}{d\eta_{34}} = -\frac{h_4 - h_3}{\eta_{34}} \quad (18a)$$

(the optimised work goes down with an increase of second-stage efficiency);

$$\lambda = r \frac{dW}{Rdr} = -\kappa \frac{T_{4s}}{h_4 - h_3} = \frac{T_{4s}}{\eta_{34}} \quad (18b)$$

(the optimised work goes up with the pressure ratio);

$$\begin{aligned} \mu &= \frac{dW}{d\varepsilon} = \left[1 - \frac{\lambda}{T_3} - \kappa \frac{1 - \eta_{34}}{h_4 - h_3} \right] (h_2 - h_1) = (h_2 - h_1) \left[1 - \frac{T_{4s}}{\eta_{34} T_3} + \frac{1 - \eta_{34}}{\eta_{34}} \right] \\ &= -(h_2 - h_1) \frac{T_{4s} - T_3}{\eta_{34} T_3} \end{aligned} \quad (18c)$$

(increasing heat exchanger efficiency lowers the optimum work); and

$$\chi = \frac{dW}{d\eta_{12}} = -\lambda \frac{h_2 - h_1}{T_{2s}} = -\frac{h_2 - h_1}{\eta_{34}} \frac{T_{4s}}{T_{2s}} \quad (18d)$$

(increasing the compressor efficiency in the first stage decreases the optimum work).

Finally the optimisation condition is

$$\frac{\eta_{12}}{\eta_{34}} \frac{T_{4s}}{T_{2s}} = 1 + \frac{1 - \varepsilon}{\eta_{34} T_3} (T_{4s} - T_3) \quad (19)$$

There is no simple analytical solution to find the optimum conditions, however, because it involves the temperature T_3 and the isentropic temperatures, which are not given directly by the η , ε parameters expressed in enthalpies. Nevertheless, the special case of the perfect heat exchanger, where $\varepsilon = 1$, leads to a somewhat simpler optimum, since we then have $\eta_{12} T_{4s} = \eta_{34} T_{2s}$ and also $T_3 = T_1$. If, in addition, the two stage efficiencies are equal, then not only does $T_{4s} = T_{2s}$, but $h_{2s} = h_{4s}$ and $h_3 = h_1$. Then using the enthalpy definitions of efficiency leads to the original optimum condition following equation (12):

$$\text{at } \varepsilon = 1 \text{ and } \eta_{12} = \eta_{34}, \text{ minimum work requires } T_2 = T_4 \quad (20)$$

that is, the stage exit temperatures are equal. Since the stage entry temperatures and the stage efficiencies are also equal, this also means that the *stage pressure ratios* are equal at an optimum.

This result duplicates the perfect gas result in the ideal gas model, but only under the ideal heat exchanger assumption. For equal efficiencies and a perfect heat exchanger, that is, not only are the optimum stage exit temperatures equal but so are the stage pressure ratios. When this specialisation is relaxed, the less explicit result of equation (19) can be employed.

That is, if the stage efficiencies are not the same, then this case with $\varepsilon = 1$ can still be treated directly. Tables of reduced pressure ratios² for ideal gases are available [3], so that from the common ambient starting temperature we can evaluate the stage pressure ratio from $T_{4s}/T_{2s} = \eta_{34}/\eta_{12}$ such that the combined ratio yield the required duty, r/r_0 .

One approach for the completely general case is as follows:

- Step 1. From the required duty and an estimate of the pressure loss ratio in the heat exchanger, find an approximate compressor pressure ratio, $\tilde{r}_{12} = \sqrt{r/r_0}$. Use the ideal gas tables for the medium together with an ideal heat exchanger and no-loss compressors to estimate $\tilde{T}_{2s} = T_2 = T_{2s}$ (as above).
- Step 2. With assumed values of η_{12} and the (given) ambient temperature T_1 , find $h_{2s}(\tilde{T}_{2s})$, $h_1(T_1)$ and hence h_2 . Then, for given ε , find h_3 and T_3 . From the second-stage duty, $r_{34} = r/r_0 r_{12}$, h_3 and reduced pressure tables, find T_{34} .
- Step 3. Evaluate equation (19) and iterate steps 2 and 3 on assumed \tilde{T}_{2s} until satisfied. Use η_{34} to find $h_4(T_4)$ and finally T_4 .

If we reimpose the perfect gas model, then equation (19) can be simplified, thus allowing for different stage efficiencies. We obtain

$$\rho_{12}^2 = \frac{\rho}{\rho_0} \frac{\eta_{12} - (1 - \varepsilon)}{\eta_{34} - (1 - \varepsilon)} \quad (21)$$

which is indeed seen to reproduce the original results when the stage efficiencies are equal. It is, however, notable that no minimum exists if, for example, the numerator or the denominator on the right-hand side of equation (21) were, separately, to go to zero or to change sign. Such values are atypical and we may say that the design is then entering an anomalous regime in which the minimum is at an imaginary stage pressure ratio. This lack of a physical mathematical minimum does not, of course, mean that there is not a 'best' value but simply that there is no vanishing of the first derivative in the optimising problem – Pontryagin's lemma [5]. This is a case where the lowest value is not a mathematical minimum but has met a constraint.

² Essentially these tabulate $r_{ref}(T) = \exp\left(\frac{1}{R} \int_{T_{ref}}^T \frac{dh}{T}\right)$ for different gases.

Commentary

The optimisation in the perfect gas case calls for equal pressure ratios in the two stages when the stage efficiencies are equal. In this model, where performance depends only on ratios of pressures and of temperatures, it would make little sense to consider different stage efficiencies. So the result can be considered as an example of an optimisation where, if the stage duties vary, one will be better than the other and the duties must be equated, whether or not the heat exchanger is perfect.

In the ideal gas generalisation, the same is true for the case with a perfect heat exchanger and equal stage compressor efficiencies. If the exchanger remains perfect and the efficiencies differ (which again is not realistic) it is possible to determine the balance of the stage duties from the ratio of stage efficiencies. When the heat exchanger effectiveness is not 100%, then equation (19) provides a less ready determination of the optimum conditions.

But if we reimpose the perfect gas approximation at this stage to obtain equation (21), then differing stage efficiencies are more believable and the optimum ratio of stage efficiencies is affected by differing stage efficiencies modified by heat exchanger effectiveness. It is seen that the first-stage duty should be increased over that of the second stage if the first-stage efficiency is higher. The anomalous point comes when $1 - \varepsilon = \eta_{12}$ or, *separately*, $1 - \varepsilon = \eta_{34}$ – not in themselves typical values. This point calls for one stage to carry all the duty and is the limit of the domain in which intercooling is advantageous. It represents the least value of compressor work in the intercooled process, whereas a further pressure ratio giving a mathematical minimum is not physically realisable (imaginary pressure ratio). The difference is between the true engineering desire to find a least value and the formal mathematical minimum, which is not necessarily physically realisable.

For the general case of an ideal gas, it seems that there is no simple algebraic condition for optimisation (although an optimum would lie close to a perfect gas approximation) and that the iteration proposed of equation (19) would not be noticeably easier than a direct numerical search for the least work. Given, however, that, for the perfect heat exchanger, the ideal gas calls for equal output temperatures and equal stage pressure ratios, we can be guided towards the best value. Perhaps the most useful piece of this analysis is the specification of the Lagrange multipliers, which directly show the designer the changes in optimum work brought about by small changes in the design pressure ratio, the heat exchanger effectiveness and the separate stage efficiencies. Bellman's principle allows a generalisation from the two-stage intercooled compression studied here to multi-stage optimisation.

The Appendix shows the development for the ideal gas with efficiencies and effectiveness defined in terms of temperature instead of enthalpy.

Acknowledgement

I am grateful to my colleague Geoff Parks for drawing my attention to this problem. The perfect gas results have been used as teaching material in the Cambridge Tripos for several years.

References

- [1] J. D. Lewins, 'The ideal gas Joule cycle at maximum specific work', *Physics Education* (Pune, India, 2000), pp. 345–353.
- [2] J. D. Lewins, 'The idea gas Joule cycle at maximum specific work', *Proc. Inst. Mech. Eng.*, **214**(Part C) (2000), 1545–1551.
- [3] M. J. Moran and H. N. Shapiro, *Fundamentals of Engineering Thermodynamics*, 3rd edition (Wiley, Chichester, 1995).
- [4] J. D. Lewins, 'Modular theory of simple heat exchangers', *Int. J. Mech. Engr. Educ.*, **17**(4) (1990), 255.
- [5] J. Lewins and A. L. Babb, 'Optimal reactor control theory', *Advances in Nuclear Science and Technology*, **4** (1968), pp. 252–308.
- [6] J. D. Lewins, 'Introducing the Lagrange multiplier to engineering mathematics', *Int. J. Mech. Engr. Educ.*, **22**(3) (1994), 191–207.

Appendix

The alternative development of the ideal gas optimisation in terms of temperatures rather than enthalpies is given below:

$$\kappa = -\frac{c_{p4}}{\eta_{34}} \quad \lambda = \frac{c_{p4}}{c_{p4s}} \frac{T_{4s}}{\eta_{34}} \quad \chi = -\frac{c_{p4}}{c_{p4s}} \frac{c_{p2s}}{\eta_{34}} \frac{T_{4s}}{T_{2s}}$$

$$\mu = \left[c_{p3} \left(1 - \frac{c_{p4}}{c_{p4s}} \frac{T_{4s}}{\eta_{34} T_3} \right) + \frac{c_{p4}}{\eta_{34}} (1 - \eta_{34}) \right] (T_2 - T_1)$$

and the optimum condition

$$\frac{\eta_{12}}{\eta_{34}} \frac{c_{p4} c_{p2s}}{c_{p4s} c_{p2}} \frac{T_{4s}}{T_{2s}} = 1 + \frac{1 - \varepsilon}{\eta_{34}} \left[\frac{c_{p4}}{c_{p2}} \left(\frac{c_{p3}}{c_{p4s}} \frac{T_{4s}}{T_3} - 1 \right) - \frac{\eta_{34}}{c_{p2}} (c_{p4} - c_{p3}) \right]$$

This is seen to be more complicated than the enthalpy form, equation (19), and the definitions in temperature are anyway less apparent than in enthalpies. The equilibrium condition does not compute as readily in the ideal gas case (although reducing to the same result in the perfect gas case) since specific heat capacities at different temperatures would be obtained by numerical differentiation of enthalpy values, better suited therefore to the enthalpy form.

A solution algorithm using tables of reduced pressure ratio against temperature would be as follows. Guess $\tilde{r}_{12}(T_1)$: $\rightarrow \tilde{T}_{2s}$ from tables; $\eta_{12} \rightarrow \tilde{T}_2$; $\varepsilon \rightarrow \tilde{T}_3$; $\tilde{r}_{34} = r/r_0$; $\tilde{r}_{12} \rightarrow \tilde{T}_{4s}$; $\eta_{34} \rightarrow \tilde{T}_4$. All specific heat capacities can be determined allowing the left-hand side and right-hand side to be compared. Iterate on \tilde{r}_{12} to equality. This seems tedious and not readily computerised unless full enthalpy data are available to look up.