Analysis of frictional forces on a rigid body

H. P. LEE, Associate Professor, Department of Mechanical and Production Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260. ⟨mpeleehp@nus.edu.sg⟩

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To analyse the motion of a rigid body sliding along a rough surface, information on the point of application of the resulting frictional and normal forces is usually not required. The present study proceeds further, by finding the direction and point of application of the resulting frictional and normal forces on a rigid body in planar motion. These findings are usually omitted in traditional analyses of these problems. It will be shown that such a sliding motion, which is assumed to be feasible with the finding of the acceleration, is not possible for certain combinations of coefficient of friction and geometry of the rigid body. Some interesting findings are highlighted and discussed.

Key words: rigid body, friction dynamics

INTRODUCTION

If we pose this question to any student: 'What is the direction of frictional force?' They will most likely reply that a frictional force always opposes the motion. This notion has been deeply embedded in students due to the way we have been teaching the subject of friction. Examples on the effect of friction in particle dynamics or rigid body dynamics are invariably in the form of a solid block sliding along a flat surface, or sliding down or up a slope under the action of gravity, or the application of an external force (Fig. 1). A block is used in these examples, as the orientation of the block remains unchanged during the motion and it can be treated just like a particle. Many textbooks on dynamics [1–4] treat this problem under the topic of 'Kinetics of Particles' and not 'Kinetics of Rigid Body', although a block is a rigid body and not a particle. If we examine the free body diagram of a typical example (Fig. 2), the normal force N and the frictional force f are usually drawn to act at a point directly below the centre of mass of a uniform rectangular block. The point of application of the resultant normal force and frictional force acting on the block is not an issue in these examples. The reason is that the acceleration of the block can be obtained by just balancing the external and kinetic force (namely the term ma) in two orthogonal directions, usually along the direction of motion and the transverse direction. Information on the point of application of the frictional and normal forces is not required for finding the acceleration. Analysis is usually terminated with the finding of the acceleration and it is not required to apply Newton's law for the balance of external moments and moment due to kinetic force. Many students therefore do no realize that the point of application of these resultant normal and frictional forces

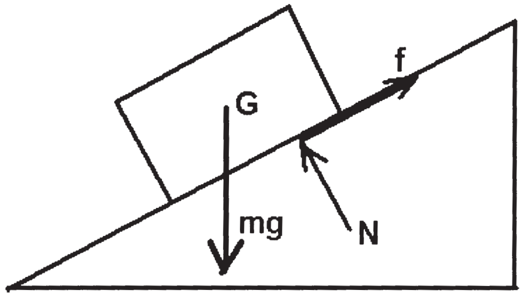

Fig. 1. A block sliding down a slope.

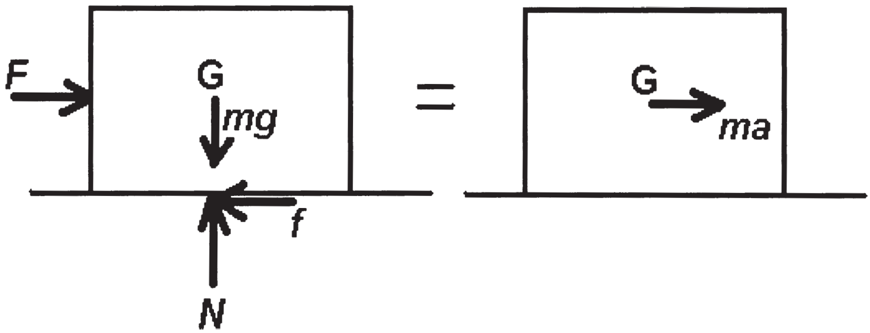

Fig. 2. A block sliding along a flat surface.

acting on the block, cannot be directly below the centre of mass or fixed arbitrarily at any point along the surface of contact. It will violate Newton's second law if this point of application of the normal and frictional forces is assigned arbitrarily. It will be shown in the present study that such a sliding motion is not possible for certain combinations of coefficient of friction and geometry of the block. The traditional practice of not checking the point of application of the frictional and normal forces may have resulted in obtaining the acceleration of the block for a sliding motion that is not possible.

A rare example pointing to the fact that the point of application of the normal force and frictional force acting on a block cannot be arbitrarily fixed, was shown in the earlier edition of the textbook by Hibbeler [5]. In that example, it was staged that the point of application must be within the span of the block, or must fall within the line of contact with the surface without further elaboration. This issue and other interesting findings will be presented and discussed in the following examples for a rigid body in planar motion.

For a rigid body in translation, the students are not concerned with the question of whether the contact of the rigid body with the surface is a point contact or a line contact (in the context of a planar motion; strictly speaking, it should be stated as line contact or surface contact). There is no ambiguity for deciding the direction of frictional force. They can easily determine the resultant acceleration of the rigid body by applying the notion that frictional force always opposes the motion. However, for a general planar motion involving both translation and rotation, the point of contact could have zero velocity. A well-known example is the motion of a disk rolling without sliding on a rough surface. Examples are also presented in an attempt to help the students to understand the effect of friction under these circumstances for planar motion.

A RECTANGULAR BLOCK SLIDING ALONG A ROUGH SURFACE

Let us examine a familiar example of a block sliding on a rough surface acted upon by a force F (Fig. 3). The point G is the centre of mass of the rigid body. The rigid body is assumed to be homogenous with the centre of mass at the centre of the block. The quantity a is the acceleration of the centre of mass. The force F is acting at a point of height h from the surface. The resultant frictional force f and normal force N are assumed to act at a point at a horizontal distance x from the centre of mass. The width of the block is $2b$ and the height is $2e$. The quantities b and e are thus, respectively, half the width and height of the block.

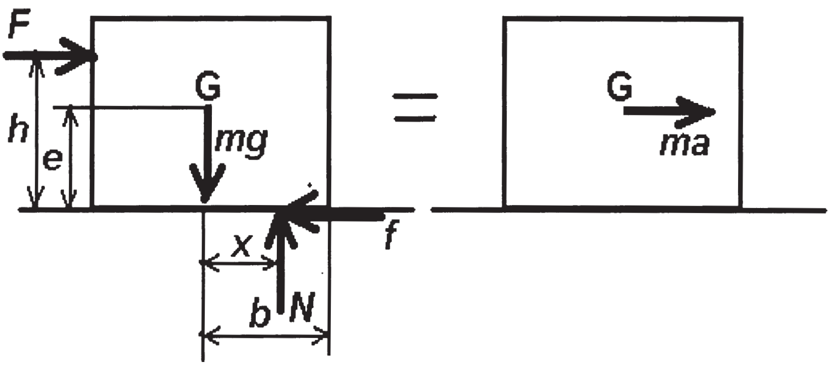

Fig. 3. A block sliding along a rough surface.

A typical question is to find the acceleration of the block. This can be done easily by balancing the external force and kinetic force ma in the horizontal, as well as the vertical directions, by applying the D'Alembert's principle.

Balance of external and kinetic forces in the vertical direction,

$$mg - N = 0$$

Therefore,

$$N = mg$$

The direction of motion of every point of the rigid body is apparently in the forward direction, the frictional force f therefore points backward, opposite to F. Balance of external and kinetic forces in the horizontal direction,

$$F - f = ma$$

However, we know that

$$f = \mu N = \mu mg$$

where μ is the coefficient of kinetic friction. Therefore,

$$a = (F - \mu mg)/m$$

Most discussions of this example in a typical textbook will stop at this point with the finding of the acceleration. The acceleration has been found and there does not seem to be any other issues left. Let us proceed one step further to determine the value of x.

Taking moment about G for the balance of external moment and moment of the kinetic force,

$$F(h - e) + \mu Ne - Nx = 0$$

or

$$F(h - e) + \mu mge - mgx = 0$$

$$x = (F/mg)(h - e) + \mu e$$

For the condition that $F > \mu mg$ of the motion to be possible, the conclusion is that the resultant frictional force and the normal force do not always act at a point directly below the centre of mass. Moreover, the width of the block $2b$ does not appear explicitly in the expression. There are some interesting issues. Let us examine the following cases in greater detail.

Case 1: $x = 0$

For $x = 0$, the point of application of the frictional and normal force is directly below the centre of mass of the block. Therefore

$$(F/mg)(h - e) + \mu e = 0$$

$$F = (mg\mu e)/(e - h)$$

As F must be positive, it is only possible if $e - h > 0$, due to the fact that the expression for the numerator is always positive. Therefore, the force F with the magnitude given by the expression must act at a point at a height below the centre of mass of the block.

Case 2: $x = b$

For $x = b$, the point of application of the normal and frictional force is at the right-hand, lower corner of the block. This can be viewed as the critical case for the impending motion of the block to tip forward about this corner, under the action of the applied force F.

$$(F/mg)(h - e) + \mu e = b$$

$$F = mg(b - \mu e)/(h - e)$$

This is the critical value of the force F for the block to tip forward. We need to examine whether this motion is possible. There are two separate cases.

Case (i) for $h - e > 0$, that is, the force F acts at a point at a height above the centre of mass of the block. For F to be positive, $b - \mu e > 0$ or $\mu < b/e$. That is, the width of the block $2b$ must be larger than $2\mu e$ for this motion to be possible.

Case (ii) for $h - e < 0$, that is, the force F acts at a point at a height below the centre of mass of the block. For F to be positive, $b - \mu e < 0$. That is, the width of the block $2b$ must be smaller than $2\mu e$ for this motion to be possible.

Case 3: $x = -b$

For $x = -b$, the point of application of the normal and frictional force is at the left-hand, lower corner of the block. This can be viewed as the critical case for the impending motion of the block, to tip backward about this corner under the action of the applied force F.

$$(F/mg)(h - e) + \mu e = -b$$

$$F = mg(b + \mu e)/(e - h)$$

This is the critical value of the force F for the block to tip backward. We also need to examine whether this motion is possible. There are also two separate cases.

Case (i) for $e - h > 0$, that is, the force F acts at a point at a point at a height below the centre of mass of the block. The term F is positive, the motion is possible.

Case (ii) for $e - h < 0$, that is, the force F acts at a point at a height above the centre of mass of the block. The term F is negative, the motion is therefore not possible.

Case 4: $0 < x < b$

For $0 < x < b$, the point of application of the normal and frictional force is at a point between the centre of the line of contact with the rough surface to the right-hand lower corner of the block.

$$0 < (F/mg)(h - e) + \mu e < b$$

There are also two separate cases.

Case (i) for $h - e > 0$, that is the force F acts at a point at a height above the centre of mass of the block.

$$-mg\mu e/(h - e) < F < mg(b - \mu e)/(h - e)$$

The left condition is automatically satisfied. As F is positive, the right condition would require that $b > \mu e$.

When $F = mg(b - \mu e)/(h - e)$, it will be the same as Case 2 when $x = b$.

Case (ii) for $h - e < 0$, that is, the force F acts at a point at a height below the centre of mass of the block.

$$-mg\mu e/(h - e) > F > mg(b - \mu e)/(h - e)$$

or

$$mg(\mu e - b)/(h - e) < F < mg(b - \mu e)/(h - e)$$

The left condition is automatically satisfied if $b > \mu e$. If $b < \mu e$, F will be bound by two positive limits. The lower limit of $F = mg(b - \mu e)/(h - e)$ is the same as Case 2 when $x = b$. The upper limit is $F = mg\mu e/(e - h)$.

Case 5: $-b < x < 0$

For $-b < x < 0$, the point of application of the normal and frictional force is at a point between the centre of the line of contact to the left-hand lower corner of the block.

$$-b < (F/mg)(h - e) + \mu e < 0$$

There are also two separate cases.

Case (i) for $h - e > 0$, that is, the force F acts at a point at a height above the centre of mass of the block.

$$mg(-b - \mu e)/(h - e) < F < -mg\mu e/(h - e)$$

The right condition cannot be satisfied as F is positive. Therefore this situation is not possible.

Case (ii) for $h - e < 0$, that is, the force F acts at a point at a height below the centre of mass of the block.

$$mg(-b - \mu e)/(h - e) > F > -mg\mu e/(h - e)$$

or

$$mg(b + \mu e)/(e - h) > F > mg\mu e/(e - h)$$

F will be bound by two positive limits. The upper limit of $F = mg(b + \mu e)/(e - h)$ is the same as Case 3 when $x = -b$. The lower limit is $F = mg\mu e/(e - h)$, same as the upper limit for Case 4.

Case 6: *h = e*

This is the case when the force F acts at the same height as the centre of mass of the block.

$$x = \mu e$$

For sliding motion,

$$-b < \mu e < b$$

Since the coefficient of friction is always greater than zero, it is not possible for $x < 0$. The remaining condition is

$$\mu < b/e$$

For the critical case with the point of application at the right-hand lower corner of the block,

$$\mu = b/e$$

For $x = 0$, $\mu = 0$.

Besides the above conditions for the six cases that have been elaborated, one should not forget that $F > \mu mg$ for a to be positive, namely for the motion to be possible.

A RECTANGULAR BLOCK SLIDING DOWN A SLOPE UNDER ITS OWN WEIGHT

The second example looks at the motion of a block sliding down as rough surface under its own weight (Fig. 4). The kinetic force (ma) is now shown in the diagram. A typical question is to determine the acceleration a of the block. The angle of inclination of the slope is θ.

The balance of external and kinetic force would result in the following expression:

$$mg \sin \theta - f = ma$$

$$mg \cos \theta = N$$

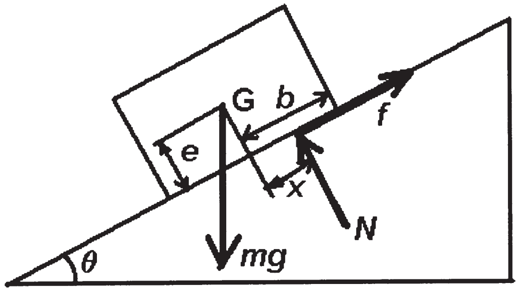

Fig. 4. A block sliding down a rough surface.

Moreover, $f = \mu N$. Therefore,

$$a = (g/m)\,(\sin\theta - \mu\cos\theta)$$

The acceleration of the block can be determined easily without knowing the value of x, the location of the point of application of the frictional force, and the normal force. Let us proceed further to locate this point.

Taking moment about G, the centre of mass

$$Nx + fe = 0$$

Therefore $x = -fe/N = -\mu e$.

Since the coefficient of friction is a positive quantity, this implies that x is a negative quantity. The point of application of the normal and frictional forces cannot be at the centre of the line of contact with the surface unless the coefficient of friction is zero. For

$$x = -b, \quad \mu = b/e$$

$$-b < x < 0, \quad \mu < b/e$$

The assumed sliding motion is not possible for the coefficient of friction greater than this value. Besides these conditions, one should not forget that $\sin\theta > \mu\cos\theta$, or $\mu < \tan\theta$, for the acceleration to be positive, or in other words, for the motion to be possible.

A DISK ROLLING WITHOUT SLIDING ON A ROUGH SURFACE ACTED UPON BY A FORCE

The next example is the motion of a disk rolling on a flat surface acted upon by a force F (Fig. 5). The radius of the disk is r. I is the moment of inertia of the disk about the centre of mass. The quantity a is the acceleration of the centre of mass of the disk. This example can be seen frequently in a typical textbook on the kinetics of a rolling disk. The reason for putting this example here is to illustrate that the direction of friction can be treated as an unknown if the velocity of the point of contact is not known.

When there is no sliding, we know that the point of contact with the surface is the instantaneous centre of rotation. The velocity is zero. We therefore could not decide which direction is the direction of frictional force f. Let f be pointing in the forward direction. Balance of forces in the horizontal direction:

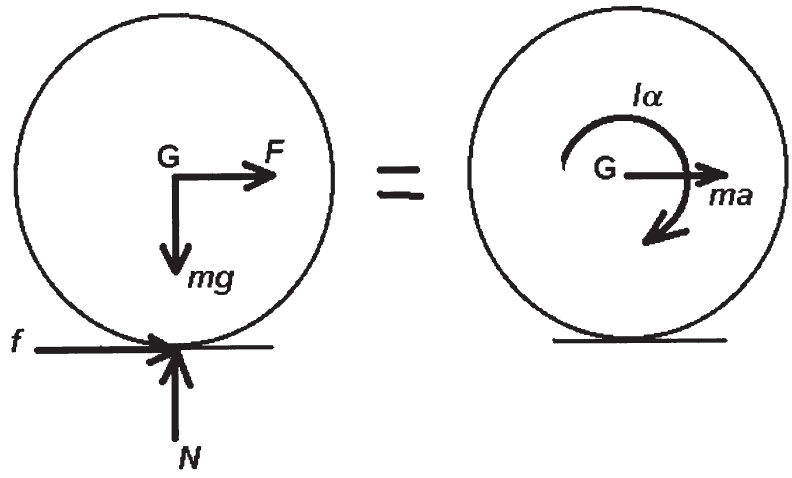

Fig. 5. A disk rolling without sliding on a flat surface acted upon by a force.

$$F + f = ma$$

Balance of forces in the vertical direction:

$$N = mg$$

Taking moment about G,

$$-fr = I\alpha$$

For rolling without sliding,

$$a = r\alpha$$

Therefore

$$f = -I(a/r)$$

We can see that the frictional force must point in the backward direction for the disk to accelerate forward. Therefore, the additional equation obtained by balancing the external moment, moment of the kinetic force and kinetic torque, will enable us to determine the direction of the frictional force. However, when there is sliding, the angular acceleration and the velocity of the centre of mass of the disk are no longer related. The direction of frictional force can be decided by the direction of the velocity of the point of contract. It is important to note that the direction of the frictional force for this case cannot be assigned arbitrarily at the beginning of the analysis.

A DISK ROLLING WITHOUT SLIDING ON A ROUGH SURFACE ACTED UPON BY A COUPLE

Let us examine the case when the disk is subjected to a couple of torque *M* (Fig. 6). If the disk rolls without sliding, once again the direction of friction is unknown. Let us assume it to be positive in the forward direction.

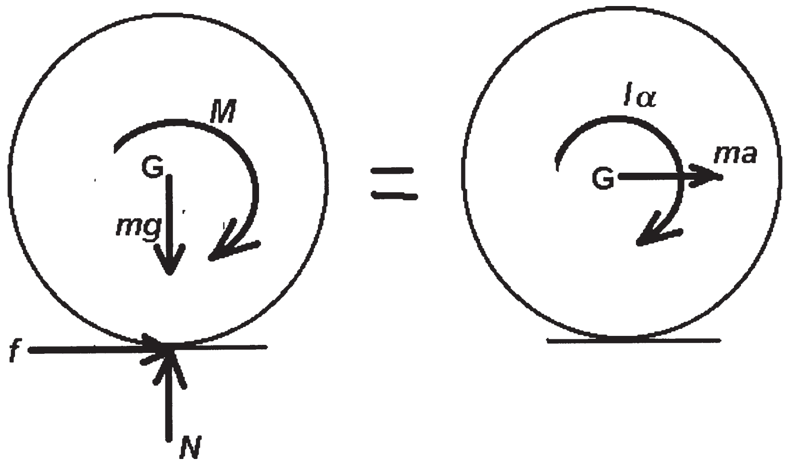

Fig. 6. A disk rolling without sliding on a flat surface acted upon by a couple.

Balance of forces in the horizontal direction:

$$f = ma$$

The term *a* is the acceleration of the centre of mass of the disk. We can see that the frictional force actually provides the force for driving the disk forward. Therefore without friction, the disk cannot move forward. The frictional force points in the forward direction. If we have assigned *f* to be positive in the backward direction, the above equation would result in a negative value for *f* to indicate that it is in the forward direction.

Taking moment about G:

$$M - fe = I\alpha = I(a/r)$$

$$M - mar = I(a/r)$$

$$a(I/r + mr) = M$$

The acceleration for the centre of mass *a* can then be computed.

Once again if there is sliding, the angular acceleration and the velocity of the centre of mass of the disk are no longer related. The direction of frictional force can be decided by the direction of the velocity of the point of contact. It is important to note that the direction of the frictional force for this case cannot be assigned arbitrarily at the beginning of the analysis.

CONCLUSION

In this paper, several examples are presented to illustrate the effect of friction for the planar motion of rigid body. In the example, for a rectangular block of width 2*b* and height 2*e*, sliding on a rough surface subjected to a force *F*, acting at a height *h*, from the surface, the findings regarding the magnitude of *F* and the point of application of the resultant frictional and normal force, *x* can be summarized as follows:

	$x = -b$	$-b < x < 0$	$x = 0$	$0 < x < b$	$x = b$
$h > e$	Not possible	Not possible	Not possible	$b > \mu e$ or $\mu < b/e$ $F < mg(b - \mu e)/(h - e)$	$b > \mu e$ or $\mu < b/e$ $F < mg(b - \mu e)/(h - e)$
$h = e$	Not possible	Not possible	$\mu = 0$	$\mu < b/e$	$\mu = b/e$
$h < e$	$F = mg(b + \mu e)/(e - h)$	$mg(b + \mu e)/(e - h) > F > mg\mu e/(e - h)$	$F = (mg\mu e)/(e - h)$	$mg(\mu e - b)/(e - h) < F < mg\mu e/(e - h)$	$b < \mu e$ or $\mu > b/e$ $F = mg(\mu e - b)/(e - h)$

Besides satisfying the above conditions, F must also be greater than μmg for the motion to be possible.

For a rectangular block of width $2b$ and height $2e$ sliding down a slope, the point of application of the frictional and normal forces cannot be at the centre of the line of contact unless the coefficient of friction is zero. Moreover, the coefficient must be less than b/e for the sliding motion to be possible. Besides these conditions, one should not forget that $\sin\theta > \mu\cos\theta$ or $\mu < \tan\theta$ for the acceleration to be positive, or in other words, for the motion to be possible.

For a point contact with zero relative velocity between a rolling disk and a rough surface, the direction of the frictional force cannot be pre-determined and can be assigned arbitrarily at the beginning of the analysis. The direction of the frictional force will be dependent on both the geometry and the external forces and moments acting on the rigid body. However, if there is sliding between the surfaces, the direction of the frictional force acting on the rigid body is always opposite to the velocity of the point of contact.

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